

# On Holomorphic Jet Bundles

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## Introduction

In this article we provide a more detailed discussion (see [W4]) of the jet bundles introduced by Green and Griffiths [G-G]. In section 1 some basic facts about these jet bundles (which are different from the usual jet bundles used in analysis) are established with the most important one being a Theorem of Green and Griffiths concerning the natural filtration of the sheaf, denoted  $\mathcal{J}_k^m X$ , of  $k$ -jet differentials of weight  $m$ . With this result many computations (of Chern classes) and properties of  $\mathcal{J}_k^m X$  can be obtained or inferred from the more familiar objects  $\odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X$  (satisfying the condition  $i_1 + 2i_2 = \dots + ki_k = m$ ). The calculation of Chern classes of  $\mathcal{J}_k^m X$  are carried out in section 1 for curves and in section 2 for surfaces. These are needed later in applying the Riemann-Roch Theorem for  $\mathcal{J}_k^m X$  and its corresponding line sheaf,  $\mathcal{O}_{\mathbf{P}(J^k X)}(m)$ , over the projectivized bundle  $\mathbf{P}(J^k X)$ . There are some complications in working with these sheaves due to the fact that the fibers of  $\mathbf{P}(J^k X)$  are weighted projective spaces and hence not smooth and moreover, the natural sheaves  $\mathcal{O}_{\mathbf{P}(J^k X)}(m)$  are not necessarily locally free if  $m$  is not divisible by  $k!$ . These minor difficulties are clarified in section 3 and is readily seen to be rather harmless. In section 4 we consider the case of surfaces of general type and here there is another complication due to the fact that, as oppose to the bundles  $\odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X$ , the sheaves  $\mathcal{J}_k^m X$  are not semi-stable (with respect to the canonical bundle of  $X$ ). This difficulty, however, can be overcome rather easily and as a result we obtain applications in the theory of holomorphic curves in surfaces of general type (hypersurfaces in  $\mathbf{P}^3$  in particular). These are presented in section 4. We also include two appendices. In appendix A the lemma of logarithmic derivatives and a version of Schwarz Lemma are presented (see [L], [L-Y], [DSW1], [DSW2], [S-Y], [W3] and [J]). Some combinatorics related to the symmetric groups which we used in the computation of Chern classes (this comes in, for example, in counting the number of positive integer solutions of the equation  $i_1 + 2i_2 + \dots + ki_k = m$ ) are presented in appendix B. For

higher dimensional manifolds the approach of Nevanlinna Theory appears to work better (see [W5]). Nevanlinna Theory for symmetric and exterior products of the cotangent bundle can be found in [St].

## § 1 Holomorphic Jet Bundles

We examine two concepts of "jet bundles" of a complex manifold. The first is the jet bundles used by analysts (PDE) and also by Faltings in his work on rational points of an ample subvariety of an abelian variety and integral points of complement of an ample divisor of an abelian variety [F]. The second is the jet bundles introduced by Green and Griffiths [G-G]. The first notion of jet bundle shall henceforth be referred to as the *full jet bundle* while the second notion of jet bundle shall be referred to as the *restricted jet bundle*. The reason for these terminologies is that the fiber dimension of the full jet bundle is much larger than that of the restricted jet bundle.

For a complex manifold  $X$  the (locally free) sheaf of germs of holomorphic tangent vector fields (differential operators of order 1) of  $X$  shall be denoted by  $T^1X$  or simply  $TX$ . An element of  $T^1X$  acts on the sheaf of germs of holomorphic functions by differentiation:

$$(D, f) \in T^1X \times \mathcal{O}_X \mapsto Df \in \mathcal{O}_X$$

and the action is linear over the complex number field  $\mathbf{C}$ , i.e.,

$$D \in \mathcal{H}om_{\mathbf{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

This concept can be extended as follows:

**Definition 1.1** Let  $X$  be a complex manifold of dimension  $n$  the sheaf of germs of holomorphic  $k$ -jets (differential operators of order  $k$ ), denoted  $\mathcal{T}^kX$ , is the subsheaf of the sheaf of homomorphisms  $\mathcal{H}om_{\mathbf{C}}(\mathcal{O}_X, \mathcal{O}_X)$  consisting of elements (differential operators) of the form

$$\sum_{j=1}^k \sum_{i_j \in \mathbf{N}} D_{i_1} \circ \dots \circ D_{i_j}$$

where  $D_{i_j} \in T^1X$ . In terms of holomorphic coordinates  $z_1, \dots, z_n$  an element of  $\mathcal{T}^kX$  is expressed as:

$$\sum_{j=1}^k \sum_{1 \leq i_1 \leq \dots \leq i_j \leq n} a_{i_1, \dots, i_j} \frac{\partial^j}{\partial z_{i_1} \dots \partial z_{i_j}}$$

where the coefficients  $a_{i_1, \dots, i_j}$  are holomorphic functions. We can also drop the requirement that the indices be non-decreasing by requiring symmetry in the coefficients, in other words, the elements of  $\mathcal{T}^k X$  can also be expressed as:

$$\sum_{j=1}^k \sum_{1 \leq i_1, \dots, i_j \leq n} a_{i_1, \dots, i_j} \frac{\partial^j}{\partial z_{i_1} \dots \partial z_{i_j}}$$

where the coefficients  $a_{i_1, \dots, i_j}$  are symmetric in the indices  $i_1, \dots, i_j$ , i.e., if  $\sigma$  is an element of the symmetric group of  $j$  elements then

$$a_{i_{\sigma(1)}, \dots, i_{\sigma(j)}} = a_{i_1, \dots, i_j}.$$

The effect of holomorphic change of coordinates from  $z = (z_1, \dots, z_n)$  to  $w = (w_1, \dots, w_n)$  is given by the transition function (for  $k = 2$ ):

$$\begin{pmatrix} (\frac{\partial}{\partial z_i})_{1 \leq i \leq n} \\ (\frac{\partial^2}{\partial z_i \partial z_k})_{1 \leq i \leq k \leq n} \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} (\frac{\partial}{\partial w_j})_{1 \leq j \leq n} \\ (\frac{\partial^2}{\partial w_j \partial w_l})_{1 \leq j \leq l \leq n} \end{pmatrix} \quad (1)$$

where  $A$  is the  $n$  by  $n$  matrix:

$$A = (\frac{\partial w_j}{\partial z_i})_{1 \leq i, j \leq n},$$

while  $B$  is the  $C_2^{n+1}$  by  $n$  matrix:

$$B = (\frac{\partial^2 w_j}{\partial z_i \partial z_k})_{1 \leq i \leq k \leq n}$$

$C$  is the  $C_2^{n+1}$  by  $C_2^{n+1}$  matrix:

$$C = (\frac{\partial w_j}{\partial z_i} \frac{\partial w_l}{\partial z_k})_{1 \leq i \leq k \leq n, 1 \leq j \leq l \leq n}$$

and  $0 = 0_{n \times C_2^{n+1}}$  is the  $n$  by  $C_2^{n+1}$  zero-matrix (here  $C_2^{n+1} = (n+1)!/(n-1)!2!$  is the usual binomial coefficient). Note that the matrix  $A$  is the transition function for the tangent bundle  $TX$  while the matrix  $C$  is the transition function of  $\odot^2 TX$ , the 2-fold symmetric product of the tangent bundle. For general  $k$  the transition function of  $T^k X$  is of the form:

$$\begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 \\ * & A_2 & 0 & 0 & 0 & 0 \\ * & * & . & 0 & 0 & 0 \\ * & * & * & . & 0 & 0 \\ * & * & * & * & . & 0 \\ * & * & * & * & * & A_k \end{pmatrix}$$

where the  $C_j^{n+j-1}$  by  $C_j^{n+j-1}$  matrix  $A_j$  is the transition function of the bundle  $\odot^j TX$ , the  $j$ -fold symmetric product of the tangent bundle. Here  $C_j^{n+j-1} = (n+j-1)!/j!(n-1)!$  is the usual binomial coefficient.

The linear (and invertible) nature of the transition functions implies that  $T^k X$  is locally free. This can also be seen by observing that  $T^{k-1} X$  injects into  $T^k X$  and there is an exact sequence of sheaves:

$$0 \rightarrow T^{k-1} X \rightarrow T^k X \rightarrow T^k X / T^{k-1} X \rightarrow 0 \quad (2)$$

where

$$T^k X / T^{k-1} X \cong \odot^k T^1 X \quad (3)$$

is the sheaf of germs of  $k$ -fold symmetric product of  $T^1 X$ , i.e., sheaf of germs of operators of the form:

$$\sum_{1 \leq i_1 \leq \dots \leq i_j \leq n} a_{i_1, \dots, i_j} \frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}}.$$

These exact sequences imply, by induction, that  $T^k X$  is locally free as the sheaves  $\odot^k T^1 X$ , being the symmetric product of the tangent sheaf, is locally free. We include here the proof of the isomorphism (3).

**Proposition 1.2** *With the notations above we have:*

$$T^k X / T^{k-1} X \cong \odot^k TX$$

where  $\odot TX$  is the  $k$ -fold symmetric product of the tangent bundle.

*Proof.* We shall define a surjection from the  $k$ -fold tensor product of  $TX$  onto the quotient  $T^k X / T^{k-1} X$ :

$$\mu : \otimes^k TX \rightarrow T^k X / T^{k-1} X$$

and then show that the surjection factors through the symmetric product resulting in a bijection. The map  $\mu$  is defined by:

$$\mu(D_1 \otimes \dots \otimes D_k) = [D_1 \circ \dots \circ D_k]$$

where  $D_i$  is (the germ of) a vector field and  $[\ ] : T^k X \rightarrow T^k X / T^{k-1} X$  is the quotient map. By definition the map  $\mu$  is surjective. To see that the map

factors through to the symmetric product it is sufficient to show that the map is invariant by any transposition, i.e.,

$$\mu(D_1 \otimes \dots \otimes D_i \otimes D_{i+1} \otimes \dots \otimes D_k) = \mu(D_1 \otimes \dots \otimes D_{i+1} \otimes D_i \otimes \dots \otimes D_k).$$

This follows from the fact that the Lie bracket  $D_i \circ D_{i+1} - D_{i+1} \circ D_i$  of the vector fields  $D_i$  and  $D_{i+1}$  is again a vector field and not a 2-jet. Thus we have:

$$D_1 \circ \dots \circ (D_i \circ D_{i+1} - D_{i+1} \circ D_i) \circ \dots \circ D_k \in T^{k-1}X$$

which implies that the map  $\mu$  descends to the symmetric product  $\odot^k TX$ . More precisely, if we denote the symmetrization operator by  $\sigma_k$  then

$$\bar{\mu}(D_1 \odot \dots \odot D_k) = \bar{\mu}(\sigma_k(D_1 \otimes \dots \otimes D_k)) \stackrel{\text{def}}{=} \mu(D_1 \otimes \dots \otimes D_k)$$

is well-defined. It is clear that  $\bar{\mu}$  is surjective and it remains to show that  $\bar{\mu} : \odot^k TX \rightarrow T^k X / T^{k-1}X$  is injective.

Let  $(z_1, \dots, z_n)$  be a local coordinate near a point  $x \in X$  then

$$\frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}} = \frac{\partial}{\partial x_{i_1}} \odot \dots \odot \frac{\partial}{\partial x_{i_k}}, \quad 1 \leq i_1 \leq \dots \leq i_k \leq n$$

is a basis of  $\odot^k TX$  at the point  $x \in X$ . If  $\bar{\mu}$  is not injective then there exists a differential operator  $\Psi$  of the form

$$\Psi = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} a_{i_1 \dots i_k} \frac{\partial^k}{\partial x_{i_1} \dots \partial x_{i_k}} - \Phi$$

where  $\Phi$  is a differential operator of order at most  $k-1$  and such that  $\bar{\mu}(\Psi) = 0$ . Apply the operator  $\Psi$  to the function  $f = x_{i_1} \dots x_{i_k}$  shows that this is possible only if all the coefficients  $a_{i_1 \dots i_k}$  are zero. This shows that  $\bar{\mu}$  is injective and completes the proof of the Proposition. QED

The restricted  $k$ -jet bundles are introduced by Green-Griffiths in [G-G]. It is defined as follows. Denote by  $\mathcal{H}_x, x \in X$  the sheaf of germs of holomorphic curves  $\{f : \Delta_r \rightarrow X, f(0) = x\}$ . Define, for  $k \in \mathbf{N}$ , an equivalence relation as follows. Let  $z_1, \dots, z_n$  be holomorphic coordinates near  $x$  and for  $f \in \mathcal{H}_x$  let  $f_i = z_i \circ f$ . Two elements  $f, g \in \mathcal{H}_x$  are said to be  $k$ -equivalent, denoted  $f \sim_k g$ , if  $f_j^{(p)}(0) = g_j^{(p)}(0)$  for all  $1 \leq p \leq k$ . The sheaf of restricted  $k$ -jets is defined to be  $J^k X = \cup_{x \in X} \mathcal{H}_x / \sim_k$ . Elements of  $J^k X$

will be denoted by  $j^k f(0) = (f(0), f'(0), \dots, f^{(k)}(0))$ . It is also clear that  $J^1 X = T^1 X = TX$  is the tangent bundle.

Note that the definition above does not depend on the choices of the coordinates near  $x$ ; for if  $(z_j \circ f)^{(p)}(0) = (z_j \circ g)^{(p)}(0)$  for  $0 \leq p \leq k$  then  $(w_j \circ f)^{(p)}(0) = (w_j \circ g)^{(p)}(0)$  for any other coordinates  $w_1, \dots, w_n$ . The effect of change of coordinates is as follows:

$$\begin{aligned} (w_j \circ f)' &= \sum_{i=1}^n \frac{\partial w_j}{\partial z_i}(f)(z_i \circ f)', \\ (w_j \circ f)'' &= \sum_{i=1}^n \frac{\partial w_j}{\partial z_i}(f)(z_i \circ f)'' + \sum_{i,k=1}^n \frac{\partial^2 w_j}{\partial z_i \partial z_k}(f)(z_i \circ f)'(z_k \circ f)' \end{aligned}$$

and for general  $k$  we have

$$(w_j \circ f)^{(k)} = \sum_{i=1}^n \frac{\partial w_j}{\partial z_i}(f)(z_i \circ f)^{(k)} + P\left(\frac{\partial^l w_j}{\partial z_{i_1} \dots \partial z_{i_l}}(f), (w_j \circ f)^{(l)}\right) \quad (4)$$

where  $P$  is a polynomial with integer coefficients in  $\partial^l w_j / \partial z_{i_1} \dots \partial z_{i_l}, (w_j \circ f)^{(l)}$  for  $j = 1, \dots, n$  and  $l = 1, \dots, k$ .

Note that the quadratic nature in  $(z_i \circ f)'$  in the formula for  $(w_i \circ f)''$  means that the sheaf  $J^k X$  is not locally free. It is instructive to compare this with the matrix  $C$  in the transition formula (1) for  $T^k X$  under the change of coordinates. In that case the formula is quadratic in the partial derivatives  $\partial w_j / \partial z_i$  but linear in  $(z_i \circ f)'$  hence the transformation can still be represented as a linear transformation while this is not the case for  $J^k X$ . There is however, a natural  $\mathbf{C}^*$ -action on  $J^k X$  defined via parametrization. Namely, for  $\lambda \in \mathbf{C}^*$  and  $f \in \mathcal{H}_x$  a map  $f_\lambda \in \mathcal{H}_x$  is defined by  $f_\lambda(t) = f(\lambda t)$ ; then  $j^k f_\lambda(0) = (f_\lambda(0), f'_\lambda(0), \dots, f^{(k)}_\lambda(0)) = (f(0), \lambda f'(0), \dots, \lambda^k f^{(k)}(0))$ . In other words the  $\mathbf{C}^*$ -action is given by

$$\lambda \cdot j^k f(0) = (f(0), \lambda f'(0), \dots, \lambda^k f^{(k)}(0)). \quad (5)$$

**Definition 1.3** The restricted  $k$ -jet bundle is defined to be  $J^k X$  together with the  $\mathbf{C}^*$ -action defined above and shall simply be denoted by  $J^k X$ .

Another difference between the full and restricted  $k$ -jet bundles is that there is, in general, no natural way of injecting  $J^{k-1} X$  into  $J^k X$ . For instance, the coordinates transformations shows that

$$(f(0), (f'_1(0), \dots, f'_n(0))) \mapsto (f(0), (f'_1(0), \dots, f'_n(0)), (0, \dots, 0))$$

is not a well-defined map of  $J^1X$  into  $J^2X$  as the condition  $f''(0) = 0$  is not preserved by a general change of coordinates. On the other hand, the transformation formulas show that  $\{j^2f = (f(0), f'(0), f''(0)) \in J^2X \mid f'(0) = 0\}$ , more generally,

$$Z_0 = \{j^k f = (f(0), f'(0), \dots, f^{(k)}(0)) \in J^kX \mid f'(0) = 0\} \quad (6)$$

is a well-defined subvariety of  $J^kX$  as the condition  $f'(0)$  is invariant under change of coordinates. Moreover, the transformation law actually says that even though the condition  $f''(0) = 0$  is coordinate dependent the conditions that  $f'(0) = f''(0) = 0$  are independent of choices of coordinates, in other words, the zero-section of  $J^2X$ , more generally, the zero-section of  $J^kX$ :

$$\{j^k f(0) \in J^kX \mid f'(0) = f''(0) = \dots = f^{(k)}(0) = 0\} \quad (7)$$

is well-defined.

**Theorem 1.4** *Let  $X$  be a complex manifold of dimension  $n$  then  $T^kX$  is a holomorphic vector bundle of rank  $r = n + C_2^{n+1} + C_3^{n+2} + \dots + C_k^{n+k-1} = \sum_{i=1}^k C_i^{n+i-1}$  while  $J^kX$  is a holomorphic  $\mathbf{C}^*$ -bundle of rank  $r = kn$  and the zero-section of  $J^kX$  is well-defined.*

As noted above there is no natural inclusion map from  $J^{k-1}X$  into  $J^kX$  there is however a natural projection map

$$p_{kj} : J^kX \rightarrow J^jX$$

for any  $j \leq k$  defined simply by

$$p_{kj}(j^k f(0)) = j^j f(0). \quad (8)$$

The projection map clearly respect the  $\mathbf{C}^*$ -action defined by (5) and so is a  $\mathbf{C}^*$ -bundle morphism.

If  $\Phi : X \rightarrow Y$  is a holomorphic map between the complex manifolds  $X$  and  $Y$  then the usual differential  $\Phi_* : T^1X \rightarrow T^1Y$  is defined. The same is true for the  $k$ -jets as the  $k$ -th order differential  $\Phi_{k*} : T^kX \rightarrow T^kY$  can be defined by

$$\Phi_{k*} = (D_1 \circ \dots \circ D_k)(g) \stackrel{\text{def}}{=} D_1 \circ \dots \circ D_k(g \circ \Phi)$$

where  $g \in \mathcal{O}_Y$ . The  $k$ -th order differential, denoted  $J^k\Phi : J^kX \rightarrow J^kY$  can also be defined:

$$J^k\Phi(j^kf(0)) \stackrel{\text{def}}{=} (\Phi \circ f)^{(k)}(0)$$

for any  $j^kf(0) \in J^kX$ . For the restricted jet bundle  $J^kX$  there is another notion closely related to (but not the same) the differential: the natural lifting of a holomorphic curve. Namely, given any holomorphic map  $f : \Delta_r \rightarrow X$  ( $0 < r \leq \infty$ ), the lifting  $j^kf : \Delta_{r/2} \rightarrow J^kX$  is defined by:

$$j^kf(\zeta) = j^kg(0), \quad \zeta \in \Delta_{r/2}$$

where  $g(\xi) = f(\zeta + \xi)$  is holomorphic for  $\xi \in \Delta_{r/2}$ .

Consider the special case  $\dim X = 1$  then  $T^kX$  and  $J^kX$  have the same rank and the underlying space of  $T^kX$  and  $J^kX$  are the same but the structures are different. Consider the map (for simplicity we write this out only for  $k = 2$ ):

$$(f(0), f'(0), f''(0)) \mapsto f''(0) \frac{\partial}{\partial z} + (f'(0))^2 \frac{\partial^2}{\partial z^2}. \quad (9)$$

which is clearly holomorphic but is not biholomorphic. For if

$$g(t) = f(0) - f'(0)t + f''(0)t^2/2$$

then  $j^2g(0) = (f(0), -f'(0), f''(0))$  and under the identification above  $j^2f(0)$  and  $j^2g(0)$  are mapped onto the same element. Moreover the map is a  $\mathbf{C}^*$ -bundle map because  $\lambda \cdot (f(0), f'(0), f''(0)) = (f(0), \lambda f'(0), \lambda^2 f''(0))$  is mapped onto the element

$$\lambda^2 \{ f''(0) \frac{\partial}{\partial z} + (f'(0))^2 \frac{\partial^2}{\partial z^2} \}.$$

More generally, for  $X$  of arbitrary dimension, the map

$$(f(0), f'(0), f''(0)) \mapsto \sum_{i=1}^n f''_i(0) \frac{\partial}{\partial z_i} + \sum_{1 \leq i, j \leq n} f'_i(0) f'_j(0) \frac{\partial^2}{\partial z_i \partial z_j} \quad (10)$$

is a holomorphic  $\mathbf{C}^*$ -bundle map from  $J^2X$  onto a  $\mathbf{C}^*$  sub-bundle of  $T^2X$ . We have already seen the case of  $n = 1$ ; for  $n = 2$  the second sum above has 3 terms:

$$(f'_1)^2, (f'_2)^2, f'_1 f'_2.$$



Thus if  $j^2 f(0)$  and  $j^2 g(0)$  have the same image then

$$(f'_1)^2 = (g'_1)^2, (f'_2)^2 = (g'_2)^2, f'_1 f'_2 = g'_1 g'_2$$

so that  $f'_1 = \pm g'_1, f'_2 = \pm g'_2$ . This means that the map is generically 2 to 1 onto its image and ramified along the subvariety  $Z_0$  defined by (6).

Returning to the case of a Riemann surface  $X$  we define a map  $p_3 : J^3 X \rightarrow T^3 X$  by the formula:

$$p_3(j^3 f(0)) = f^{(3)}(0) \frac{\partial}{\partial z} + f''(0) f'(0) \frac{\partial^2}{\partial z^2} + (f'(0))^3 \frac{\partial^3}{\partial z^3}$$

and in general  $p_k : J^k X \rightarrow T^k X$  by the formula:

$$p_k(j^k f(0)) = \sum_{j=1}^k f^{(j)}(0) (f'(0))^{k-j} \frac{\partial^{k-j+1}}{\partial z^{k-j+1}}.$$

For the higher dimensional manifold  $X$  the maps are defined by

$$\begin{aligned} p_3(j^3 f(0)) &= \sum_{i=1}^n f_i^{(3)}(0) \frac{\partial}{\partial z_i} + \sum_{i,j=1}^n f_i''(0) f_j'(0) \frac{\partial^2}{\partial z_i \partial z_j} + \\ &+ \sum_{i,j,k=1}^n f_i'(0) f_j'(0) f_k'(0) \frac{\partial^3}{\partial z_i \partial z_j \partial z_k} \end{aligned}$$

and in general  $p_k : J^k X \rightarrow T^k X$  by the formula:

$$p_k(j^k f(0)) = \sum_{j=1}^k \sum_{i=1}^n \{f_i^{(j)}(0) \prod_{i_1, \dots, i_{k-j} \neq i} f_{i_l}'(0)\} \frac{\partial^{k-j+1}}{\partial z_i \partial z_{i_1} \dots \partial z_{i_{k-j}}}. \quad (11)$$

It is clear from the definition of the map  $p_k$  that:

**Theorem 1.5** *Let  $J^k X$  and  $T^k X$  be, respectively, the restricted and the full  $k$ -jet bundles over a complex manifold  $X$ . Then the map defined by (10) is a holomorphic  $\mathbf{C}^*$ -bundle map which is generically finite to 1 onto its image. Moreover, the map is ramified precisely along the subvariety  $Z_0 = \{j^k f(0) \in J^k X \mid f'(0) = 0\}$ .*

We consider now the "dual" of the jet bundles.

**Definition 1.6** The dual of the full jet bundles  $T^k X$  shall be referred to as the sheaf of germs of  $k$ -jet forms and shall be denoted by  $T_k^* X$ . The global sections shall be referred to as  $k$ -jet forms. For  $m \in \mathbf{N}$  the  $m$ -fold symmetric product shall be denoted by either  $\odot^m T_k^* X$  and its global sections shall be referred to as  $k$ -jet forms of weight  $m$ .

By definition, a  $k$ -jet form of weight  $m$  assigns to each point  $x \in X$  a homogeneous (with respect to the standard  $\mathbf{C}^*$ -action of  $T^k X$  as a vector bundle) polynomial of degree  $m$  on the fiber  $T_x^k X$  (where  $T^k X$  is the  $k$ -jet bundle). Let  $(U, z_1, \dots, z_n)$  be a local holomorphic coordinates over  $U$  then

$$\begin{aligned} (e_i &= \frac{\partial}{\partial z_i})_{1 \leq i \leq n}, \\ (e_{i_1 i_2} &= \frac{\partial^2}{\partial z_{i_1} \partial z_{i_2}})_{1 \leq i_1 \leq i_2 \leq n}, \\ &\cdot \\ &\cdot \\ &\cdot \\ (e_{i_1 \dots i_k} &= \frac{\partial^k}{\partial z_{i_1} \dots \partial z_{i_k}})_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} \end{aligned}$$

is a basis of  $T^k X|_U$ . The dual basis shall be denoted, formally, by

$$\begin{aligned} (e_i^* &= dz_i)_{1 \leq i \leq n}, \\ (e_{i_1 i_2}^* &= d^2 z_{i_1} z_{i_2})_{1 \leq i_1 \leq i_2 \leq n}, \\ &\cdot \\ &\cdot \\ &\cdot \\ (e_{i_1 \dots i_k}^* &= d^k z_{i_1} \dots z_{i_k})_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n}. \end{aligned}$$

An element of  $T_k^* X$  is then of the form:

$$\omega = \sum_{j=1}^k \sum_{1 \leq i_1 \leq \dots \leq i_j \leq n} a_{i_1, \dots, i_j} e_{i_1, \dots, i_j}^*$$

where the coefficients  $a_{i_1, \dots, i_j}$  are holomorphic functions. Sometimes it is convenient to express the sum without the restriction as in the second sum

above but insisting on the symmetry of the coefficients (see also definition 1.1):

$$\omega = \sum_{j=1}^k \sum_{1 \leq i_1, \dots, i_j \leq n} a_{i_1, \dots, i_j} e_{i_1, \dots, i_j}^*$$

where the coefficients  $a_{i_1, \dots, i_j}$  are symmetric in the indices. We can also write down a basis for the symmetric product  $\odot^m T^k X$  and its dual basis for  $\odot^m T_k^* X$ . It is convenient to use the following notations and conventions for the index set. Let

$$\mathcal{I}_k = \{I = (i_1, \dots, i_k) \mid i_k \in \mathbf{N} \cup \{0\}, 0 \leq i_1 \leq \dots \leq i_k \leq n \text{ and not all } i_j = 0\}$$

be endowed with lexicographical order and let

$$\mathcal{J}_m(\mathcal{I}_k) = \{J = (I_1, \dots, I_m) \mid I_j \in \mathcal{I}_k, I_1 \leq \dots \leq I_m\}.$$

With these notations, for example, the basis for  $T_k^* X$  over  $U$  is simply expressed as  $B_k^* = \{e_I^* = e_{i_1}^* \odot \dots \odot e_{i_k}^* \mid I \in \mathcal{I}_k\}$  with the conventions that  $e_0 = 1$ . Analogously, a basis for  $\odot^m T_k^* X$  over  $U$  is expressed as  $B_k^{*m} = \{e_J^* = e_{I_1}^* \odot \dots \odot e_{I_m}^* \mid J \in \mathcal{J}_m(\mathcal{I}_k)\}$ . Moreover, a section  $\omega \in H^0(U, \odot^m T_k^* X)$  is expressed as

$$\omega = \sum_{J \in \mathcal{J}_m(\mathcal{I}_k)} a_J e_J^*$$

where the coefficients are holomorphic functions on  $U$ .

Taking the dual of the sequence (2) we get an exact sequence:

$$0 \rightarrow \odot^k T_1^* X \rightarrow T_k^* X \rightarrow T_{k-1}^* X \rightarrow 0. \quad (12)$$

For example, for  $k = 3$  we have two exact sequences:

$$0 \rightarrow \odot^3 T_1^* X \rightarrow T_3^* X \rightarrow T_2^* X \rightarrow 0,$$

$$0 \rightarrow \odot^2 T_1^* X \rightarrow T_2^* X \rightarrow T_1^* X \rightarrow 0.$$

In particular, by Whitney's Formula:

$$c_1(T_3^* X) = c_1(T_2^* X) + c_1(\odot^3 T_1^* X) = c_1(T_1^* X) + c_1(\odot^2 T_1^* X) + c_1(\odot^3 T_1^* X).$$

In general, we have, by induction:

**Theorem 1.7** *The first Chern number of the bundle of  $k$ -jet forms is given by the formula:*

$$c_1(T_k^*X) = \sum_{j=1}^k c_1(\odot^j T_1^*X).$$

*In particular, if  $X$  is a Riemann surface then*

$$c_1(T_k^*X) = \sum_{j=1}^k j c_1(T_1^*X) = \frac{k(k+1)}{2} c_1(\mathcal{K}_X)$$

*where  $\mathcal{K}_X = T_1^*X$  is the canonical bundle of  $X$ .*

Note that if  $X$  is a Riemann surface then the rank of  $T_k^*X$  is  $k$ .

**Corollary 1.8** *Let  $X$  be a projective manifold and suppose that the cotangent bundle  $T_1^*X$  is ample then  $T_k^*X$  is ample for all  $k$ .*

**Definition 1.9** The dual of  $J^kX$ , i.e., germs of  $\omega : j^kX|_U \rightarrow \mathbf{C}$  such that  $\omega(\lambda \cdot j^k f) = \lambda^m \omega(j^k f)$  for some positive integer  $m$ , shall be referred to as the sheaf of germs of  $k$ -jet differentials and shall be denoted by  $\mathcal{J}_k^*X$ . A jet differential  $\omega$  satisfying the homogeneity above with integer  $m$  is said to be a  $k$ -jet differential of weight  $m$ . The sheaf of  $k$ -jet differential of weight  $m$  shall be denoted by  $\mathcal{J}_k^mX$ .

It follows from the definition of the  $\mathbf{C}^*$ -action on  $J^kX$  that a  $k$ -jet differential  $\omega$  of weight  $m$  is of the form:

$$\omega(j^k f) = \sum_{|I_1|+2|I_2|+\dots+k|I_k|=m} a_{I_1,\dots,I_k}(z) (f')^{I_1} \dots (f^{(k)})^{I_k} \quad (13)$$

where  $a_{I_1,\dots,I_k}$  are holomorphic functions,  $I_j = (i_{1j}, \dots, i_{nj})$ ,  $n = \dim X$  are the multi-indices with each  $i_{lj}$  being a non-negative integer and  $|I_j| = i_{1j} + \dots + i_{nj}$ . In terms of a local coordinate  $(z_1, \dots, z_n)$  near a point  $z$ ,

$$(f')^{I_1} \dots (f^{(k)})^{I_k} = (f_1')^{i_{11}} \dots (f_n')^{i_{n1}} \dots (f_1^{(k)})^{i_{1k}} \dots (f_n^{(k)})^{i_{nk}}.$$

Moreover the coefficients  $a_{I_1,\dots,I_k}(z)$  are symmetric with respect to the indices in each  $I_j$ . More precisely,

$$a_{(i_{\sigma_1(1)1}, \dots, i_{\sigma_1(n)1}), \dots, (i_{\sigma_k(1)k}, \dots, i_{\sigma_k(n)k})} = a_{(i_{11}, \dots, i_{n1}), \dots, (i_{1k}, \dots, i_{nk})}$$

where each  $\sigma_j : \{1, \dots, n\} \rightarrow \{1, \dots, n\}, j = 1, \dots, n$  is a permutation of  $n$ -elements.

Let  $\mathcal{L}_k^m X$  be the subsheaf of  $\mathcal{J}_k^m X$  consisting of elements of the form:

$$\omega(j^k f) = \sum_{j=1}^k \sum_{|I_1|+j|I_j|=m} a_{I_1}(f)(f')^{I_1}(f^{(j)})^{I_j}. \quad (14)$$

Note that the coefficient of  $(f')^{I_1}(f^{(j)})^{I_j}$  depends only on  $I_1$  but is independent of  $I_j$ . This sheaf shall be referred to as jet differentials of linear type.

**Lemma 1.10** *The sheaf  $\mathcal{L}_k^m X$  of jet differentials of linear type is well-defined. For  $m = k = 2$  we have  $\mathcal{L}_2^2 X = \mathcal{J}_2^2 X$  and if  $X$  is a Riemann surface then  $\mathcal{L}_3^3 X = \mathcal{J}_3^3 X$ .*

*Proof.* The change of variable formulas (4) shows that a jet differential of the form (14) is invariant by change of coordinates. QED

There is a differentiatial operation  $d : \mathcal{J}_k^m X \rightarrow \mathcal{J}_{k+1}^{m+1} X$  naturally defined by:

$$d\omega(j^{k+1} f) \stackrel{\text{def}}{=} (\omega(j^k f))'. \quad (15)$$

It should be noted that in contrast to exterior differentiation of forms  $d \circ d \neq 0$  on jet differentials. In particular, given a holomorphic function  $\phi$  defined on some open neighborhood  $U$ ,

$$d^{(k)}\phi(j^k f) = (\phi \circ f)^{(k)} \quad (16)$$

which is a non-trivial  $k$ -jet differential for general  $\phi$ . Another difference between jet differentials and exterior differential forms is that a lower order jet differential can be naturally identified with a jet differential of higher order. More precisely, the natural projection  $p_{kj} : J^k X \rightarrow J^j X$  defined for  $k \geq j$  induces an injection  $p_{kj}^* : \mathcal{J}_j^m X \rightarrow \mathcal{J}_k^m X$  defined naturally by:

$$p_{kj}^* \omega(j^k f) \stackrel{\text{def}}{=} \omega(p_{kj}(j^k f)) = \omega(j^j f). \quad (17)$$

We shall simply write  $\omega(j^k f) = \omega(j^j f)$  if no confusion arises. Moreover, the symmetric product of a  $k$ -jet differential of weight  $m$  and a  $k'$ -jet differential of weight  $m'$  is a  $(k + k')$ -jet differential of weight  $m + m'$ .

Consider first the case of  $k = 2$  and denote by

$$p : J^2 X \rightarrow p(J^2 X) \subset T^2 X$$

the generically 2 to 1 map onto its image as defined in the previous section. Let  $\omega \in H^0(U, T_2^* X)$  considered as a linear (along the fibers) functional

$$\omega : T^2 X|_U \rightarrow \mathbf{C}.$$

Consider the composite map

$$\omega \circ p : J^2 X|_U \rightarrow \mathbf{C}.$$

By the definition of  $p$  we observe that

$$\omega \circ p(\lambda \cdot j^2 f) = \omega(\lambda^2 \cdot p(j^2 f)) = \lambda^2 \omega \circ p(j^2 f)$$

is homogeneous of degree 2. In other words, the composite  $\omega \circ p$  is a section of  $\mathcal{J}_2^2 X$  over  $U$ . Thus we have a well-defined  $\mathbf{C}^*$ -bundle map

$$q = p^* : T_2^* X \rightarrow \mathcal{J}_2^2 X.$$

Consider again the special case of a Riemann surface  $X$  and  $k = 2$ . Let  $\omega \in H^0(U, T_2^* X)$  be a 2-jet form then locally  $\omega$  is simply of the form

$$\omega = a dz + b d^2 z^{(2)}$$

where  $dz$  is the dual of  $\partial/\partial z$  and  $d^2 z^{(2)}$  (the notation is formal and should not be confused with differentiating  $z^2$  twice) with  $a$  and  $b$  being local holomorphic functions. By the definition of  $p$  we have:

$$p^* \omega(j^2 f) = a(f) f'' + b(f) (f')^2. \quad (18)$$

If  $p^* \omega(j^2 f) = a(f) f'' + b(f) (f')^2 = 0$  for all  $j^2 f$  then, by taking  $j^2 f = (f(0), f'(0), f''(0))$  such that  $f'(0) = 0$  and  $f''(0) \neq 0$ , we see that  $a(f(0)) = 0$  (as  $x = f(0)$  is an arbitrary point of  $U$ , we have  $a \equiv 0$ ) and so  $p^* \omega(j^2 f) = b(f) (f')^2$  for all  $j^2 f$ . Now choosing  $j^2 f(0)$  so that  $f'(0) \neq 0$  this time shows that  $b(f(0)) = 0$ . In other words, the map  $p^* : T_2^* X \rightarrow \mathcal{J}_2^2 X$  is injective. On the other hand, any homogeneous polynomial of degree 2 of  $J_x^2 X$  is the germ of a section of the form as in (18) where  $a$  and  $b$  are holomorphic functions defined on some open neighborhood of  $x$  and that  $f(0) = x$ . This shows

that  $p^*$  is actually an isomorphism between  $T_2^*X$  and  $\mathcal{J}_2^2X$ . Analogously, a section of  $\odot^2 T^*X$  is of the form

$$\omega = b(dz \odot dz)$$

where  $b$  is a holomorphic function on  $U$ . Then the pull-back

$$p^*(\omega)(j^2 f) = b(f)(f')^2.$$

In other words

$$p^*(\odot^2 T^*X) \cong \odot^2 T^*X$$

and hence we conclude that the pull-back of the sequence:

$$0 \rightarrow \odot^2 T^*X \rightarrow T_2^*X \rightarrow T^*X \rightarrow 0$$

yields an exact sequence:

$$0 \rightarrow p^*(\odot^2 T^*X) \cong \odot^2 T^*X \rightarrow \mathcal{J}_2^2X \rightarrow \mathcal{J}_2^2X/p^*(\odot^2 T^*X) \cong T^*X \rightarrow 0.$$

The same argument works also for  $k = 3$ ; for general  $k$  an analogous argument shows that  $p_k^*(T_k^*X)$  is isomorphic to the sheaf of jet differentials of linear type  $\mathcal{L}_k^k$ :

**Theorem 1.11** *For a complex manifold  $X$  the pull-back  $p_k^*(T_k^*X)$  is  $\mathbf{C}^*$ -isomorphic to  $\mathcal{L}_k^kX$  where  $p_k : J^kX \rightarrow T^kX$  is the map defined by (11). Then  $p_k^*(T_k^*X)$  is  $\mathbf{C}^*$ -isomorphic to  $\mathcal{L}_k^kX$ .*

*Proof.* We have already seen the case  $k = 2$  and suppose now that  $\omega \in H^0(U, T_k^*X)$  is a  $k$ -jet form then locally  $\omega$  is of the form

$$\omega = a_1 dz + a_2 d^2 z^{(2)} + \dots + a_k d^k z^{(k)}$$

where  $d^j z^{(j)}$  is the dual of the differential operator  $\partial^j / \partial z^j$  and each  $a_j$  is a holomorphic function on  $U$ . Pulling back we get:

$$p_k^* \omega(j^k f) = \sum_{j=1}^k a_j f^{(j)}(f')^{k-j}$$

and suppose that  $p_k^* \omega(j^k f) \equiv 0$ . Consider first the case  $k = 3$  then for any  $x \in U$  choosing  $j^3 f$  so that  $f(0) = x, f'(0) = 0$  shows that  $a_3 f^{(3)}(0) = 0$  so  $a_3(x) = 0$ . Since  $x$  is arbitray the function  $a_3 \equiv 0$ . Thus

$$0 \equiv p_3^* \omega(j^3 f) = f' \{a_2 f'' + a_1 (f')^2\}$$

and so we have

$$a_2 f'' + a_1 (f')^2 \equiv 0$$

on  $J^2 X|_U \setminus \{f' \neq 0\}$ . Let  $\phi : \Delta_\epsilon \rightarrow \Delta_\epsilon$  be a holomorphic function such that  $\phi(0) = 0, \phi'(0) = 1$  then  $(f \circ \phi)' = f'(\phi)\phi', (f \circ \phi)'' = f''(\phi)(\phi')^2 + f'(\phi)\phi''$  and the condition that  $f'(0) \neq 0$  implies that we may choose  $\phi$  so that the condition that the first jet is non-zero, (i.e.,  $(f \circ \phi)'(0) \neq 0$ ) is preserved but

$$(f \circ \phi)'' = f''(\phi)(\phi')^2 + f'(\phi)\phi'' = 0$$

i.e., choose  $\phi$  so that  $\phi''(0) = -f''(0)/f'(0)$ . This yields:

$$a_2(\phi)(f \circ \phi)''(0) = a_2(\phi)(f \circ \phi)''(0) + a_1(\phi)((f \circ \phi)')^2(0) = 0$$

so  $a_2 \equiv 0$  (because  $x = f(0)$  is an arbitrary point) and the original equation is reduced to the equation  $a_1(f)(f')^3 \equiv 0$ . Thus by choosing  $f'(0) \neq 0$  we conclude that  $a_1(f(0)) = 0$ ; this implies that  $a_1 \equiv 0$  as well. This establishes injectivity of the map  $p_3^*$ ; surjectivity follows from the fact that an element of  $\mathcal{J}_3^3 X$  is of the form

$$a_3 f^{(3)} + a_2 f' f'' + a_1 (f')^3.$$

In general we have, by setting  $f' = 0$ , that  $a_k \equiv 0$  and then:

$$0 \equiv p_k^* \omega = f' \sum_{j=1}^{k-1} a_j f^{(j)} (f')^{k-1-j}$$

and so

$$0 \equiv \sum_{j=1}^{k-1} a_j f^{(j)} (f')^{k-1-j}$$

on  $J^k X \setminus \{f' \neq 0\}$ . This shows injectivity; surjectivity now follows from the definition of  $\mathcal{L}_k^k X$ . The proof is then completed by induction and by reparametrization. QED

The following Theorem can be found (without proof) in Green-Griffiths [G-G], we include a proof here for the sake of completeness:

**Theorem 1.12** *There exists a filtration of  $\mathcal{J}_k^m X$ :*

$$\mathcal{J}_{k-1}^m X = \mathcal{F}_k^0 \subset \mathcal{F}_k^1 \subset \dots \subset \mathcal{F}_k^{[m/k]} = \mathcal{J}_k^m X$$



(where  $[m/k]$  is the greatest integer smaller than or equal to  $m/k$ ) such that

$$\mathcal{F}_k^i / \mathcal{F}_k^{i-1} \cong \mathcal{J}_{k-1}^{m-ki} X \otimes (\odot^i T^* X).$$

*Proof.* The filtrations are defined as follows. Since a  $(k-1)$ -jet differential of weight  $m$  is also a  $k$ -jet differential of weight  $m$  thus

$$F_k^0 = \mathcal{J}_{k-1}^m X \subset \mathcal{J}_k^m X$$

which in terms of the expression (13) for jet differentials consists of elements of which does not contain any terms involving  $f^{(k)}$ ; put it another way the exponent  $I_k$  for  $f^{(k)}$  satisfies the condition  $|I_k| = 0$ :

$$\begin{aligned} \omega(j^k f) &= \sum_{|I_1|+2|I_2|+\dots+(k-1)|I_{k-1}|=m} a_{I_1,\dots,I_{k-1}}(z)(f')^{I_1}\dots(f^{(k-1)})^{I_{k-1}} \\ &= \sum_{|I_1|+2|I_2|+\dots+k|I_k|=m, |I_k|=0} a_{I_1,\dots,I_k}(z)(f')^{I_1}\dots(f^{(k)})^{I_k}. \end{aligned}$$

For any  $0 \leq j \leq [m/k]$  we define  $F_j \subset \mathcal{J}_k^m X$  to be the sheaf of germs consisting elements so that  $|I_k| \leq j$ :

$$F_k^j = \{\omega | \omega(j^k f) = \sum_{|I_1|+2|I_2|+\dots+k|I_k|=m, |I_k| \leq j} a_{I_1,\dots,I_k}(z)(f')^{I_1}\dots(f^{(k)})^{I_k}\}. \quad (19)$$

By definition, we have, for  $1 \leq j \leq [m/k]$ :

$$\begin{aligned} F_k^j / F_k^{j-1} &= \{\omega | \omega(j^k f) \\ &= \sum_{|I_1|+2|I_2|+\dots+k|I_k|=m, |I_k|=j} a_{I_1,\dots,I_k}(z)(f')^{I_1}\dots(f^{(k)})^{I_k}\} \end{aligned}$$

and the claim is that

$$F_k^j / F_k^{j-1} \cong \mathcal{J}_{k-1}^{m-kj} X \otimes (\odot^j T^* X).$$

We first establish the special case of a Riemann surface. In this case a  $k$ -jet differential of weight  $m$  is of the form

$$\omega(j^k f) = \sum_{i_1+2i_2+\dots+ki_k=m} a_{i_1,\dots,i_k}(z)((z \circ f)')^{i_1}\dots((z \circ f)^{(k)})^{i_k}$$

where we identify  $f$  with  $z$  being a local coordinate on an open coordinate neighborhood  $U \subset X$  and  $i_j$  are non-negative integers; the subsheaves  $F_k^j$  is of the form:

$$F_k^j = \{\omega | \omega(j^k f) = \sum_{i_1+2i_2+\dots+ki_k=m, i_k \leq j} a_{i_1, \dots, i_k}(z)(f')^{i_1} \dots (f^{(k)})^{i_k}\}$$

for  $0 \leq j \leq [m/k]$  and

$$F_k^j / F_k^{j-1} = \{\omega | \omega(j^k f) = \sum_{i_1+2i_2+\dots+ki_k=m, i_k=j} a_{i_1, \dots, i_k}(z)(f')^{i_1} \dots (f^{(k)})^{i_k}\}$$

for  $1 \leq j \leq [m/k]$ . We first define a map

$$L_U : F_k^j / F_k^{j-1} |_U \rightarrow \mathcal{J}_{k-1}^{m-kj} X \otimes (\odot^j T^* X) |_U$$

where

$$\begin{aligned} L_U & \left( \sum_{i_1+2i_2+\dots+ki_k=m, i_k=j} a_{i_1, \dots, i_k}(z)(f')^{i_1} \dots (f^{(k)})^{i_k} \right) \\ &= (f^{(k)})^j \sum_{i_1+2i_2+\dots+(k-1)i_{k-1}=m-kj} a_{i_1, \dots, j}(z)(f')^{i_1} \dots (f^{(k-1)})^{i_{k-1}} \end{aligned}$$

The fact that  $L_U$  is an isomorphism is clear and the fact that  $L = L_U$  (where  $\mathcal{U} = \{U\}$  is an open cover of  $X$  by coordinate neighborhoods) follows from the following observation that (see (4)) if  $(V, w)$  is another coordinate neighborhood then

$$((w \circ f)^{(k)})^j = ((\partial w / \partial z)(z \circ f)^{(k)} + P)^j = ((\partial w / \partial z)(z \circ f)^{(k)})^j + Q$$

where  $P$  and  $Q$  are polynomials in the variables  $\partial^s w_i / \partial z_l^s, 1 \leq i, l \leq n, 1 \leq s \leq k$  and in  $(z \circ f)^{(r)}, 1 \leq r \leq k-1$ . In particular,  $Q$  is a  $(k-1)$ -jet differential of total weight  $m-kj$ . In other words,

$$((w \circ f)^{(k)})^j = (\partial w / \partial z)^j ((z \circ f)^{(k)})^j \mod F_k^{j-1}$$

and the transisition function  $(\partial w / \partial z)^j$  is the same as the transistioon function for  $\odot^j T^* X$ .

The higher dimensional case is notationally more complicated but the proof is essentially the same. QED

As an immediate consequence (see Green-Griffiths [G-G]) we have:

**Corollary 1.13** *Let  $X$  be a smooth projective variety then  $\mathcal{J}_k^m X$  admits a composition series whose factors contain all bundles of the form:*

$$(\odot^{i_1} T^* X) \otimes \dots \otimes (\odot^{i_k} T^* X)$$

where  $i_j$  ranges over all non-negative integers satisfying

$$i_1 + 2i_2 + \dots + ki_k = m.$$

The first Chern number of  $c_1(\mathcal{J}_k^m X)$  is given by:

$$c_1(\mathcal{J}_k^m X) = \sum_{i_1+2i_2+\dots+ki_k=m, i_j \in \mathbf{Z}_{\geq 0}} c_1((\odot^{i_1} T^* X) \otimes \dots \otimes (\odot^{i_k} T^* X)).$$

In particular, for a curve  $X = C$ ,

$$c_1(\mathcal{J}_k^m C) = \sum_{i_1+2i_2+\dots+ki_k=m, i_j \in \mathbf{Z}_{\geq 0}} (i_1 + i_2 + \dots + i_k) c_1(T^* C).$$

The preceding Theorem can be used in calculating the Chern classes of  $\mathcal{J}_k^m X$ .

**Example 1.14** For example, for  $m = k = 2$ , the filtration is given by:

$$\odot^2 T^* X = \mathcal{J}_1^2 X = \mathcal{S}_2^0 \subset \mathcal{S}_2^1 = \mathcal{J}_2^2 X, \quad \mathcal{S}_2^1 / \mathcal{S}_2^0 \cong T^* X$$

we have the following exact sequence:

$$0 \rightarrow \odot^2 T^* X \rightarrow \mathcal{J}_2^2 X \rightarrow T^* X \rightarrow 0.$$

Thus the first Chern numbers are related by the formula:

$$c_1(\mathcal{J}_2^2 X) = c_1(\odot^2 T^* X) + c_1(T^* X).$$

The filtration of  $\mathcal{J}_3^3 X$  is as follows:

$$\mathcal{J}_3^3 X = \mathcal{S}_3^1 \supset \mathcal{S}_3^0 = \mathcal{J}_2^3 X, \quad \mathcal{J}_3^3 X / \mathcal{J}_2^3 X = \mathcal{S}_3^1 / \mathcal{S}_3^0 \cong T^* X.$$

Hence we have an exact sequence:

$$0 \rightarrow \mathcal{J}_2^3 X \rightarrow \mathcal{J}_3^3 X \rightarrow T^* X \rightarrow 0.$$

Now the filtration of  $\mathcal{J}_2^3 X$  is

$$\mathcal{J}_2^3 X = \mathcal{S}_2^1 \supset \mathcal{S}_2^0 = \mathcal{J}_1^3 X, \quad \mathcal{J}_2^3 X / \mathcal{J}_1^3 X \cong T^* X \otimes T^* X$$

and, since  $\mathcal{J}_1^3 X = \odot^3 T^* X$ , we have an exact sequence:

$$0 \rightarrow \odot^3 T^* X \rightarrow \mathcal{J}_2^3 X \rightarrow T^* X \otimes T^* X \rightarrow 0.$$

From these 2 exact sequences we get

$$c_1(\mathcal{J}_3^3 X) = c_1(T^* X) + c_1(T^* X \otimes T^* X) + c_1(\odot^3 T^* X).$$

From basic representation Theory (or just simple liner algebra in this special case) we know that  $T^* X \otimes T^* X = \odot^2 T^* X \oplus \wedge^2 T^* X$  hence,

$$c_1(\mathcal{J}_3^3 X) = c_1(T^* X) + c_1(\odot^2 T^* X) + c_1(\odot^3 T^* X) + c_1(\wedge^2 T^* X).$$

In representation theory  $\wedge^2 T^* X$  is the Weyl module  $W_{1,1}^* X$  associate to the partition  $\{1, 1\}$  (see [F-H]). Thus we have:

$$c_1(\mathcal{J}_3^3 X) = \sum_{j=1}^3 c_1(\odot^j T^* X) + c_1(W_{1,1}^* X). \quad (20)$$

In the special case of a Riemann surface  $\wedge^2 T^* X$  is the zero-sheaf. Thus for a curve we have

$$c_1(\mathcal{J}_3^3 X) = (1 + 2 + 3)c_1(T^* X) = 6c_1(T^* X).$$

For  $m = k = 4$ , we have the following filtrations:

$$\mathcal{J}_4^4 X = \mathcal{S}_4^1 \supset \mathcal{S}_4^0 = \mathcal{J}_3^4 X, \quad \mathcal{J}_4^4 X / \mathcal{J}_3^4 X = \mathcal{S}_4^1 / \mathcal{S}_4^0 \cong T^* X,$$

$$\mathcal{J}_3^4 X = \mathcal{S}_3^1 \supset \mathcal{S}_3^0 = \mathcal{J}_2^4 X, \quad \mathcal{J}_3^4 X / \mathcal{J}_2^4 X = \mathcal{S}_3^1 / \mathcal{S}_3^0 \cong T^* X \otimes T^* X,$$

and

$$\mathcal{J}_2^4 X = \mathcal{S}_2^2 \supset \mathcal{S}_2^1 \supset \mathcal{S}_2^0 = \mathcal{J}_1^4 X,$$

with

$$\mathcal{J}_2^4 X / \mathcal{S}_2^1 = \odot^2 T^* X, \quad \mathcal{S}_2^1 / \mathcal{S}_2^0 \cong T^* X \otimes (\odot^2 T^* X).$$

The exact sequences associate to the filtration for  $\mathcal{J}_4^4 X$  are:

$$0 \rightarrow \mathcal{J}_3^4 X \rightarrow \mathcal{J}_4^4 X \rightarrow T^* X \rightarrow 0;$$

$$\begin{aligned}
0 &\rightarrow \mathcal{J}_2^4 X \rightarrow \mathcal{J}_3^4 X \rightarrow T^* X \otimes T^* X \rightarrow 0; \\
0 &\rightarrow S_1 \rightarrow \mathcal{J}_2^4 X \rightarrow \odot^2 T^* X \rightarrow 0; \\
0 &\rightarrow \odot^4 T^* X \rightarrow S_1 \rightarrow T^* X \otimes (\odot^2 T^* X) \rightarrow 0.
\end{aligned}$$

Thus the Chern number formula:

$$\begin{aligned}
c_1(\mathcal{J}_4^4 X) &= c_1(T^* X) + c_1(T^* X \otimes T^* X) + c_1(\odot^2 T^* X) \\
&\quad + c_1(T^* X \otimes (\odot^2 T^* X)) + c_1(\odot^4 T^* X).
\end{aligned}$$

Note that (by elementary representation theory)

$$T^* X \otimes (\odot^k T^* X) = W_{k,1}^* X \oplus (\odot^{k+1} T^* X)$$

where  $W_{k,1}^*$  is the Weyl module associate to the partition  $\{k, 1\}$  thus:

$$c_1(\mathcal{J}_4^4 X) = c_1(\odot^2 T^* X) + \sum_{i=1}^4 c_1(\odot^i T^* X) + \sum_{i=1}^2 c_1(W_{j,1}^* X). \quad (21)$$

In particulr, if  $X$  is a curve then

$$c_1(\mathcal{J}_4^4 X) = (1 + 2 + 2 + 3 + 4)c_1(T^* X) = 12c_1(T^* X).$$

Recall that  $c_1(T_4^* X) = 10c_1(T^* X)$ .

For  $m = k = 5$ , we have the following filtrations:

$$\begin{aligned}
\mathcal{J}_5^5 X &= S_5^1 \supset S_5^0 = \mathcal{J}_4^5 X, \quad \mathcal{J}_5^5 X / \mathcal{J}_4^5 X = S_5^1 / S_5^0 \cong T^* X, \\
\mathcal{J}_4^5 X &= S_4^1 \supset S_4^0 = \mathcal{J}_3^5 X, \quad \mathcal{J}_4^5 X / \mathcal{J}_3^5 X = S_4^1 / S_4^0 \cong T^* X \otimes T^* X, \\
\mathcal{J}_3^5 X &= S_3^1 \supset S_3^0 = \mathcal{J}_2^5 X, \quad \mathcal{J}_3^5 X / \mathcal{J}_2^5 X \cong T^* X \otimes (\mathcal{J}_2^2 X), \\
\mathcal{J}_2^5 X &= S_2^2 \supset S_2^1 \supset S_2^0 = \mathcal{J}_1^5 X, \quad \mathcal{J}_2^5 X / S_2^1 = (\odot^2 T^* X) \otimes T^* X, \\
&\quad S_2^1 / S_2^0 \cong T^* X \otimes (\odot^3 T^* X).
\end{aligned}$$

The exact sequences associate to the filtration for  $\mathcal{J}_5^5 X$  are:

$$\begin{aligned}
0 &\rightarrow \mathcal{J}_4^5 X \rightarrow \mathcal{J}_5^5 X \rightarrow T^* X \rightarrow 0; \\
0 &\rightarrow \mathcal{J}_3^5 X \rightarrow \mathcal{J}_4^5 X \rightarrow T^* X \otimes T^* X \rightarrow 0; \\
0 &\rightarrow \mathcal{J}_2^5 X \rightarrow \mathcal{J}_3^5 X \rightarrow T^* X \otimes \mathcal{J}_2^2 X \rightarrow 0; \\
0 &\rightarrow \mathcal{S}_2^1 \rightarrow \mathcal{J}_2^5 X \rightarrow (\odot^2 T^* X) \otimes T^* X \rightarrow 0, \\
0 &\rightarrow \odot^5 T^* X \rightarrow \mathcal{S}_2^1 \rightarrow T^* X \otimes (\odot^3 T^* X) \rightarrow 0.
\end{aligned}$$

This yields the formula:

$$\begin{aligned}
& c_1(\mathcal{J}_5^5 X) \\
&= c_1(T^* X) + c_1(T^* X \otimes T^* X) + c_1(T^* X \otimes \mathcal{J}_2^2 X) \\
&\quad + c_1((\odot^2 T^* X) \otimes T^* X) + c_1(T^* X \otimes (\odot^3 T^* X)) + c_1(\odot^5 T^* X) \\
&= c_1(T^* X) + c_1(T^* X \otimes T^* X) + c_1(T^* X \otimes (\odot^2 T^* X)) \\
&\quad + c_1(T^* X \otimes T^* X) + c_1((\odot^2 T^* X) \otimes T^* X) \\
&\quad + c_1(T^* X \otimes (\odot^3 T^* X)) + c_1(\odot^5 T^* X)
\end{aligned}$$

where we have used the fact that

$$c_1(T^* X \otimes \mathcal{J}_2^2 X) = c_1(T^* X \otimes (\odot^2 T^* X)) + c_1(T^* X \otimes T^* X).$$

Recall that

$$\begin{aligned}
T^* X \otimes T^* X &= \odot^2 T^* X \oplus \wedge^2 T^* X, \\
T^* X \otimes (\odot^2 T^* X) &= \odot^3 T^* X \oplus W_{2,1}^* X, \\
T^* X \otimes (\odot^3 T^* X) &= \odot^4 T^* X \oplus W_{3,1}^* X
\end{aligned}$$

(in general we have

$$T^* X \otimes (\odot^d T^* X) = \odot^{d+1} T^* X \oplus W_{d,1}^* X.)$$

Thus we have:

$$\begin{aligned}
& c_1(\mathcal{J}_5^5 X) \\
&= \sum_{j=2}^3 c_1(\odot^j T^* X) + \sum_{j=1}^5 c_1(\odot^j T^* X) + \sum_{j=1}^2 c_1(W_{j,1}^* X) + \sum_{j=1}^3 c_1(W_{j,1}^* X).
\end{aligned}$$

In particulr, if  $X$  is a curve then

$$c_1(\mathcal{J}_5^5 X) = (1 + 2 + 3 + 2 + 3 + 4 + 5)c_1(T^* X) = 20c_1(T^* X).$$

For  $m = k = 6$ , we have the following filtrations:

$$\begin{aligned}
\mathcal{J}_6^6 X &= S_6^1 \supset S_6^0 = \mathcal{J}_5^6 X, \quad \mathcal{J}_6^6 X / \mathcal{J}_5^6 X = S_6^1 / S_6^0 \cong T^* X, \\
\mathcal{J}_5^6 X &= S_5^1 \supset S_5^0 = \mathcal{J}_4^6 X, \quad \mathcal{J}_5^6 X / \mathcal{J}_4^6 X = S_5^1 / S_5^0 \cong T^* X \otimes T^* X, \\
\mathcal{J}_4^6 X &= S_4^1 \supset S_4^0 = \mathcal{J}_3^6 X, \quad \mathcal{J}_4^6 X / \mathcal{J}_3^6 X \cong T^* X \otimes \mathcal{J}_2^3 X,
\end{aligned}$$

$$\mathcal{J}_3^6 X = S_3^2 \supset S_3^1 \supset S_3^0 = \mathcal{J}_2^6 X,$$

with factors

$$\begin{aligned} \mathcal{J}_3^6 X / \mathcal{S}_3^1 &= \odot^2 T^* X, \quad S_3^1 / S_3^0 \cong T^* X \otimes \mathcal{J}_2^3 X, \\ \mathcal{J}_2^6 X &= S_2^3 \supset S_2^2 \supset S_2^1 \supset S_2^0 = \odot^6 T^* X, \quad \mathcal{J}_2^6 X / \mathcal{S}_2^2 = \odot^3 T^* X, \\ S_2^2 / S_2^1 &\cong (\odot^2 T^* X) \otimes (\odot^2 T^* X), \quad S_2^1 / S_2^0 \cong T^* X \otimes (\odot^4 T^* X). \end{aligned}$$

The exact sequences associate to the filtration for  $\mathcal{J}_6^6 X$  are:

$$\begin{aligned} 0 &\rightarrow \mathcal{J}_5^6 X \rightarrow \mathcal{J}_6^6 X \rightarrow T^* X \rightarrow 0; \\ 0 &\rightarrow \mathcal{J}_4^6 X \rightarrow \mathcal{J}_5^6 X \rightarrow T^* X \otimes T^* X \rightarrow 0; \\ 0 &\rightarrow \mathcal{J}_3^6 X \rightarrow \mathcal{J}_4^6 X \rightarrow T^* X \otimes \mathcal{J}_2^3 X \rightarrow 0; \\ 0 &\rightarrow \mathcal{S}_3^1 \rightarrow \mathcal{J}_3^6 X \rightarrow \odot^2 T^* X \rightarrow 0, \\ 0 &\rightarrow \mathcal{J}_2^6 X \rightarrow \mathcal{S}_3^1 \rightarrow T^* X \otimes \mathcal{J}_2^3 X \rightarrow 0, \\ 0 &\rightarrow \mathcal{S}_2^2 \rightarrow \mathcal{J}_2^6 X \rightarrow \odot^3 T^* X \rightarrow 0, \\ 0 &\rightarrow \mathcal{S}_2^1 \rightarrow \mathcal{S}_2^2 \rightarrow (\odot^2 T^* X) \otimes (\odot^2 T^* X) \rightarrow 0, \\ 0 &\rightarrow \odot^6 T^* X \rightarrow \mathcal{S}_2^1 \rightarrow T^* X \otimes (\odot^4 T^* X) \rightarrow 0. \end{aligned}$$

This yields the formula:

$$\begin{aligned} c_1(\mathcal{J}_6^6 X) &= c_1(T^* X) + c_1(T^* X \otimes T^* X) + 2c_1(T^* X \otimes \mathcal{J}_2^3 X) \\ &\quad + c_1(\odot^2 T^* X) + c_1(\odot^3 T^* X) + c_1((\odot^2 T^* X) \otimes (\odot^2 T^* X)) \\ &\quad + c_1(T^* X \otimes (\odot^4 T^* X)) + c_1(\odot^6 T^* X) \\ &= c_1(T^* X) + 3c_1(T^* X \otimes T^* X) + 2c_1(T^* X \otimes (\odot^3 T^* X)) \\ &\quad + c_1(\odot^2 T^* X) + c_1(\odot^3 T^* X) + c_1((\odot^2 T^* X) \otimes (\odot^2 T^* X)) \\ &\quad + c_1(T^* X \otimes (\odot^4 T^* X)) + c_1(\odot^6 T^* X). \end{aligned}$$

where we have used the fact that

$$c_1(T^* X \otimes \mathcal{J}_2^3 X) = c_1(T^* X \otimes (\odot^3 T^* X)) + c_1(T^* X \otimes T^* X).$$

In particulr, if  $X$  is a curve then

$$c_1(\mathcal{J}_6^6 X) = (1 + 6 + 8 + 2 + 3 + 4 + 5 + 6)c_1(T^* X) = 35c_1(T^* X).$$

The calculation before can be carried out in a much simpler fashion as follows. A *partition* of a natural number  $m$  is a set of *positive integers*  $k_1, \dots, k_q$  such that  $m = k_1 + \dots + k_q$ . A partition can be expressed as

$$m = \sum_{j=1}^k j i_j$$

where the integers  $i_j = \#$  of  $j$ 's in  $\{k_1, \dots, k_q\}$  are non-negative. Obviously we have  $1 \leq q \leq k$  and  $1 \leq k_i \leq k$  for all  $i$ . The following result is well-known in representation theory and in combinatorics (see [H-W]):

**Theorem 1.15** *The number, denoted  $p(m)$ , of classes of  $S_m$  (the symmetric group on  $m$  elements) is equal to the number of partitions of  $m$  and also to the number of (inequivalent) irreducible representations of  $S_m$ . The number  $p(m)$  is asymptotically approximated by the formula of Hardy-Ramanujan*

$$p(m) \sim \frac{e^{\pi\sqrt{2m/3}}}{4m\sqrt{3}}.$$

**Remark 1.16** The first few numbers are as follows:

$$\begin{aligned} p(1) &= 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, p(6) = 11, p(7) = 15, \\ p(8) &= 22, p(9) = 30, p(10) = 42, p(11) = 56, p(12) = 77, p(13) = 101. \end{aligned}$$

We are interested in the case

$$k = \lambda_1 + \lambda_2 + \dots + \lambda_{\rho_\lambda}$$

where  $\lambda_1 \geq \lambda_2 \dots \geq \lambda_{\rho_\lambda} \geq 1$ . Define  $l_i = \lambda_i + \rho_\lambda - i, i = 1, \dots, \rho_\lambda$ . Then the dimension  $d_\lambda$  of the representation  $V_\lambda, \lambda = (\lambda_1, \dots, \lambda_{\rho_\lambda})$  associated to the partition  $\lambda$  is given by the formula  $d_\lambda = 1$  if  $\rho_\lambda = 1$  and for  $\rho_\lambda \neq 1$  (see [F-H], p. 50):

$$d_\lambda = \frac{k!}{l_1! \dots l_{\rho_\lambda}!} \prod_{1 \leq i < j \leq \rho_\lambda} (l_i - l_j) \quad (22)$$

The number  $\rho_\lambda$  shall be referred to as *the length of the partition  $\lambda$* .

We consider also the case of partitioning a number by a partition of fixed length  $k$ . Denote by  $p_k(m)$  the number of solutions of the equation

$$x_1 + \dots + x_k = m$$

with the condition that  $1 \leq x_k \leq x_{k-1} \leq \dots \leq x_1$ . This number is obviously equal to the number of solutions of the equation

$$y_1 + \dots + y_k = m - k$$



with the condition that  $0 \leq y_k \leq y_{k-1} \leq \dots \leq y_1$ . If there are exactly  $i$  of the integers  $y_i$  which are positive then these are the solutions of  $x_1 + \dots + x_i = m - k$  ( $x_i \leftrightarrow y_i + 1$ ) and so there are  $p_i(m - k)$  of such solutions; consequently we have:

**Theorem 1.17** *With the notations above we have*

$$p_k(m) = \sum_{i=0}^k p_i(m - k)$$

if  $1 \leq k \leq m$  and with the convention that  $p_0(0) = 1, p_0(m) = 0$  if  $m > 0$  and  $p_k(m) = 0$  if  $k > m$ .

The following identity is easily established by induction:

**Theorem 1.18** *The number  $p_k(m)$  satisfies the following recursive relation:*  
 $p_k(m) = p_{k-1}(m - 1) + p_k(m - k)$ .

Obviously we have  $p_1(m) = p_m(m) = 1$  and  $p_2(m) = m/2$  or  $(m - 1)/2$  according to  $m$  being even or odd. Thus Theorem 1.19 yields  $p_3(m) = p_2(m - 1) + p_3(m - 3)$ ,  $p_4(m) = p_3(m - 1) + p_4(m - 4)$ ,  $p_5(m) = p_4(m - 1) + p_5(m - 5)$  and we get for example

$$p_1(6) = 1, p_2(6) = 3, p_6(6) = 1$$

$$p_3(6) = p_2(5) + p_3(3) = 3,$$

$$p_4(6) = p_3(5) + p_4(2) = 2,$$

$$p_5(6) = p_4(5) + p_5(1) = 1$$

hence as  $p(m) = \sum_k p_k(m)$  we have

$$p(6) = \sum_{k=1}^6 p_k(6) = 1 + 3 + 3 + 2 + 1 + 1 = 11.$$

For  $m = 7$  we have

$$p_1(7) = 1, p_2(7) = 3, p_7(7) = 1$$

$$p_3(7) = p_2(6) + p_3(4) = p_2(6) + p_2(3) = 4,$$

$$p_4(7) = p_3(6) = 3,$$

$$\begin{aligned}
p_5(7) &= p_4(6) = 2, \\
p_6(7) &= p_5(6) = 1 \\
p(7) &= \sum_{k=1}^7 p_k(7) = 1 + 3 + 4 + 3 + 2 + 1 + 1 = 15.
\end{aligned}$$

The *total length of all partitions*  $L(m)$  of a positive integer  $m$  is defined to be

$$L(m) = \sum_{j=1}^m jp_j(m).$$

For example if  $m = 6$  then  $L(6) = 1 + 6 + 9 + 8 + 5 + 6 = 35$  and for  $m = 7$ ,  $L(7) = 1 + 6 + 12 + 12 + 10 + 6 + 7 = 54$ . More generally for  $k \leq m$

$$L_k(m) = \sum_{j=1}^k jp_j(m) \tag{23}$$

shall be referred to as the total length of partitions of  $m$  of length at most  $k$ . The following Lemma is easily established from the definitions:

**Lemma 1.19** *With the notations above we have*

$$\sum_{\lambda} \sum_{j=1}^k i_j = \sum_{\lambda} \rho_{\lambda} = \sum_{j=1}^k jp_j(m)$$

where the sum on the right is taken over all partition  $\lambda = (\lambda_1, \dots, \lambda_{\rho_{\lambda}})$  of  $m$ ,  $1 \leq \lambda_{\rho_{\lambda}} \leq \dots \leq \lambda_2 \leq \lambda_1$ ,  $\rho_{\lambda} \leq k$  and  $i_j = \#$  of  $j$ 's in  $\{\lambda_1, \dots, \lambda_{\rho_{\lambda}}\}$ .

One has the following well-known asymptotic formula:

**Theorem 1.20** *For  $m \rightarrow \infty$  the number  $p_k(m)$  is asymptotically given by:*

$$p_k(m) \sim \frac{m^{k-1}}{(k-1)!k!}.$$

The preceding discussions yield the following Theorem:

**Theorem 1.21** *Let  $X$  be a non-singular pojective curve then the Chern number of  $\mathcal{J}_k^m X$  is given by*

$$c_1(\mathcal{J}_k^m X) = L_k(m)c_1(\mathcal{K}_X) = \sum_{j=1}^k jp_j(m)c_1(\mathcal{K}_X) = \sum_{j=1}^k jp_j(m)c_1(\mathcal{K}_X)$$

where  $\mathcal{K}_X$  is the canonical bundle of  $X$ . If we fix  $k$  and let  $m \rightarrow \infty$  then asymptotically:

$$c_1(\mathcal{J}_k^m X) \sim kp_k(m) \sim \frac{m^{k-1}}{(k-1)!(k-1)!}.$$

We give as examples the explicit calculation of the above. For  $m = k = 3$ , we have  $p(3) = 3$  and the possible indices are tabulated below:

	$\lambda$	$\rho_\lambda$	$d_\lambda$	$i_1$	$i_2$	$i_3$	$\sum_{j=1}^k i_j$
1	(1, 1, 1)	3	1	3	0	0	3
2	(2, 1)	2	2	1	1	0	2
3	(3)	1	1	0	0	1	1

The Chern number of a curve  $X$  is obtained by summing the last column:

$$c_1(\mathcal{J}_3^3 X) = (1 + 2 + 3)c_1(T^*X) = 6c_1(T^*X).$$

For  $m = k = 4$ , we have  $p(4) = 5$  and the possible indices are listed below

	$\lambda$	$\rho_\lambda$	$d_\lambda$	$i_1$	$i_2$	$i_3$	$i_4$	$\sum_{j=1}^k i_j$
1	(1, 1, 1, 1)	4	1	4	0	0	0	4
2	(2, 1, 1)	3	3	2	1	0	0	3
3	(3, 1)	2	3	1	0	1	0	2
4	(2, 2)	2	2	0	2	0	0	2
5	(4)	1	1	0	0	0	1	1

The Chern number of a curve  $X$  is obtained by summing the last column:

$$c_1(\mathcal{J}_4^4 X) = 12c_1(T^*X).$$

For  $m = k = 5$ , we have  $p(5) = 7$  and the possible indices are listed below

	$\lambda$	$\rho_\lambda$	$d_\lambda$	$i_1$	$i_2$	$i_3$	$i_4$	$i_5$	$\sum_{j=1}^k i_j$
1	(1, 1, 1, 1, 1)	5	1	5	0	0	0	0	5
2	(2, 1, 1, 1)	4	4	3	1	0	0	0	4
3	(3, 1, 1)	3	6	2	0	1	0	0	3
4	(2, 2, 1)	3	5	1	2	0	0	0	3
5	(4, 1)	2	4	1	0	0	1	0	2
6	(3, 2)	2	15	0	1	1	0	0	2
7	(5)	1	1	0	0	0	0	1	1

The Chern number of a curve  $X$  is obtained by summing the last column:

$$c_1(\mathcal{J}_5^5 X) = 20c_1(T^*X).$$

For  $m = k = 6$ , we have  $p(6) = 11$  and the possible indices are listed below

	$\lambda$	$\rho_\lambda$	$d_\lambda$	$i_1$	$i_2$	$i_3$	$i_4$	$i_5$	$i_6$	$\sum_{j=1}^k i_j$
1	(1, 1, 1, 1, 1, 1)	6	1	6	0	0	0	0	0	6
2	(2, 1, 1, 1, 1)	5	5	4	1	0	0	0	0	5
3	(3, 1, 1, 1)	4	10	3	0	1	0	0	0	4
4	(2, 2, 1, 1)	4	9	2	2	0	0	0	0	4
5	(4, 1, 1)	3	10	2	0	0	1	0	0	3
6	(3, 2, 1)	3	36	1	1	1	0	0	0	3
7	(2, 2, 2)	3	5	0	3	0	0	0	0	3
8	(5, 1)	2	30	1	0	0	0	1	0	2
9	(4, 2)	2	9	0	1	0	1	0	0	2
10	(3, 3)	2	5	0	0	2	0	0	0	2
11	(6)	1	1	0	0	0	0	0	1	1

The Chern number of a curve  $X$  is obtained by summing the last column:

$$c_1(\mathcal{J}_6^6 X) = 35c_1(T^*X).$$

For  $m = k = 7$ , we have  $p(7) = 15$  and the possible indices are listed below

	$\lambda$	$\rho_\lambda$	$d_\lambda$	$i_1$	$i_2$	$i_3$	$i_4$	$i_5$	$i_6$	$i_7$	$\sum_{j=1}^k i_j$
1	(1, 1, 1, 1, 1, 1, 1)	7	1	7	0	0	0	0	0	0	7
2	(2, 1, 1, 1, 1, 1)	6	6	5	1	0	0	0	0	0	6
3	(3, 1, 1, 1, 1)	5	15	4	0	1	0	0	0	0	5
4	(2, 2, 1, 1, 1)	5	14	3	2	0	0	0	0	0	5
5	(4, 1, 1, 1)	4	20	3	0	0	1	0	0	0	4
6	(3, 2, 1, 1)	4	35	2	1	1	0	0	0	0	4
7	(2, 2, 2, 1)	4	14	1	3	0	0	0	0	0	4
8	(5, 1, 1)	3	15	2	0	0	0	1	0	0	3
9	(4, 2, 1)	3	35	1	1	0	1	0	0	0	3
10	(3, 3, 1)	3	21	1	0	2	0	0	0	0	3
11	(3, 2, 2)	3	21	0	2	1	0	0	0	0	3
12	(6, 1)	2	6	1	0	0	0	0	1	0	2
13	(5, 2)	2	14	0	1	0	0	1	0	0	2
14	(4, 3)	2	14	0	0	1	1	0	0	0	2
15	(7)	1	1	0	0	0	0	0	0	1	1

The Chern number of a curve  $X$  is obtained by summing the last column:

$$c_1(\mathcal{J}_7^7 X) = 54c_1(T^*X).$$

We list below the next few values of  $L(k)$ :

$$L(8) = 86, L(9) = 128, L(10) = 192, L(11) = 275, L(12) = 399, L(13) = 556$$

$$L(14) = 780, L(15) = 1068, L(16) = 1463.$$

## § 2 Computation of Chern Classes in Complex Surfaces

We now treat the case of a complex surface (i.e., complex dimension 2). First we establish some basic facts:

**Lemma 2.1** *Let  $X$  be a nonsingular complex surface then*

$$\begin{aligned} c_1(\odot^m T^* X) &= \frac{m(m+1)}{2} c_1(T^* X), \\ c_2(\odot^m T^* X) &= a(m) c_1^2(T^* X) + b(m) c_2(T^* X) \end{aligned}$$

where  $a(m) = m(m^2 - 1)(3m + 2)/24$ ,  $b(m) = m(m + 1)(m + 2)/6$ .

*Proof.* the case of the first Chern class is straight forward and the calculation is omitted (see section 4 for a slightly more general calculation. To compute the Chern numbers of  $\odot^2 E$  we proceed formally by writing the total Chern class  $c(E) = (1 + (\lambda_1 + \lambda_2)x + \lambda_1 \lambda_2 x^2)$  then the total Chern class of  $\odot^2 E$  is (keep in mind that  $\text{rank } \odot^2 E = 3$ ):

$$(1 + 2\lambda_1 x)(1 + 2\lambda_2 x)(1 + (\lambda_1 + \lambda_2)x)$$

and a calculation (mod  $x^3$ ) yields:

$$1 + 3(\lambda_1 + \lambda_2)x + [4\lambda_1 \lambda_2 + 2(\lambda_1 + \lambda_2)^2]x^2.$$

This shows that

$$c_1(\odot^2 E) = 3c_1(E), \quad c_2(\odot^2 E) = 2c_1^2(E) + 4c_2(E)$$

and the Lemma is verified in this case.

Next we compute the Chern numbers of  $\odot^3 E$ . With a similar formalism (and keep in mind that the rank of  $\odot^3 E$  is 4), we have:

$$\begin{aligned} c(\odot^3 E) &= (1 + 3\lambda_1 x)(1 + 3\lambda_2 x)(1 + (2\lambda_1 + \lambda_2)x)(1 + (\lambda_1 + 2\lambda_2)x) \\ &= 1 + 6(\lambda_1 + \lambda_2)x + \{11(\lambda_1 + \lambda_2)^2 + 10\lambda_1 \lambda_2\}x^2 \pmod{x^3}. \end{aligned}$$

This shows that

$$c_1(\odot^3 E) = 6c_1(E), \quad c_2(\odot^3 E) = 11c_1^2(E) + 10c_2(E).$$

For general  $m$  we observe that

$$c_1(\odot^m E) = \begin{cases} (mp_2(m) + \frac{m}{2})c_1(E), & \text{if } m \text{ is even,} \\ (mp_2(m) + m)c_1(E), & \text{if } m \text{ is odd} \end{cases}$$

where  $p_2(m)$  is the number of solutions of  $m$  with partitions of fixed length 2 defined above. The Lemma follows by recalling that  $p_2(m) = m/2$  (resp.  $(m-1)/2$ ) if  $m$  is even (resp. odd). For the tensor product we observe that

$$c_1(\otimes^m E) = \sum_{i=0}^m \frac{i + (m-i)}{2} C_i^m c_1(E)$$

which follows from the fact that a partition  $m$  of length 2 can be written simply as  $l = (l_1 = i, l_2 = m-i)$ . Previously, this was defined by requiring that  $l_1 \geq l_2 \geq 1$  but in the preceeding formula we include all partitions  $l = (l_1, l_2), l_i \geq 0, l_1 + l_2 = m$ . This accounts for the extra term  $m/2$  (resp.  $m$ ) in the formula for the symmetric product and also the factor  $1/2$  in the formula for tensor product.

The second Chern class is somewhat more complicated. Given an integer  $m$  the *non-negative* partitions of  $m$  of length 2 are  $\{(m-i, i), i = 0, \dots, m\}$  and

$$c(\odot^m E) = \prod_{i=0}^m (1 + ((m-i)\lambda_1 + i\lambda_2)x) \pmod{x^3}.$$

The coefficients of  $x^2$  is the second Chern class and is given by the following sums if  $m$  is even:

$$\begin{aligned} s_0 &= \sum_{i=0}^{\frac{m}{2}-1} ((m-i)\lambda_1 + i\lambda_2)(i\lambda_1 + (m-i)\lambda_2) \\ s_1 &= \sum_{i=1}^{m-1} \{m\lambda_1((m-i)\lambda_1 + i\lambda_2) + m\lambda_2(i\lambda_1 + (m-i)\lambda_2)\} \\ s_2 &= \sum_{i=2}^{m-2} \{((m-1)\lambda_1 + \lambda_2)((m-i)\lambda_1 + i\lambda_2) + \\ &\quad + (\lambda_1 + (m-1)\lambda_2)(i\lambda_1 + (m-i)\lambda_2)\} \end{aligned}$$

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$$\begin{aligned}
s_j &= \sum_{i=j}^{m-j} \{((m-j)\lambda_1 + j\lambda_2)((m-i)\lambda_1 + i\lambda_2) + \\
&\quad + (j\lambda_1 + (m-j)\lambda_2)(i\lambda_1 + (m-i)\lambda_2)\} \\
&\quad \dots \\
&\quad \dots \\
&\quad \dots \\
s_{(m/2)-1} &= \{(\frac{m}{2} + 1)\lambda_1 + (\frac{m}{2} - 1)\lambda_2\} \{ \frac{m}{2}\lambda_1 + \frac{m}{2}\lambda_2 \} \\
c_2(\odot^m E) &= s_0 + s_1 + \dots + s_{(m/2)-1}.
\end{aligned}$$

By simple algebra, we have

$$\begin{aligned}
s_0 &= \sum_{i=0}^{\frac{m}{2}-1} i(m-i)(\lambda_1^2 + \lambda_2^2) + \sum_{i=0}^{\frac{m}{2}-1} (i^2 + (m-i)^2)\lambda_1\lambda_2 \\
&= \sum_{i=0}^{\frac{m}{2}-1} i(m-i)(\lambda_1 + \lambda_2)^2 + \sum_{i=0}^{\frac{m}{2}-1} (m-2i)^2\lambda_1\lambda_2 \\
&= \sum_{i=0}^{\frac{m}{2}-1} i(m-i)c_1(E)^2 + \sum_{i=0}^{\frac{m}{2}-1} (m-2i)^2c_2(E).
\end{aligned}$$

For  $s_j, j \geq 1$  the main observation is that each of these can be expressed as  $(\lambda_1 + \lambda_2)^2$  and so involves only  $c_1^2$ , indeed we have, for  $1 \leq j \leq (m/2) - 1$ ,

$$\begin{aligned}
s_j &= \sum_{i=j}^{m-j} (m^2 - m(i+j) + 2ij)(\lambda_1 + \lambda_2)^2 \\
&= \sum_{i=j}^{m-j} (m^2 - m(i+j) + 2ij)c_1(E)^2.
\end{aligned}$$

If  $m$  is odd:

$$s_0 = \sum_{i=0}^{\frac{m-1}{2}} ((m-i)\lambda_1 + i\lambda_2)(i\lambda_1 + (m-i)\lambda_2)$$



and  $s_j, j = 1, \dots, \frac{m-1}{2}$  are defined as before with  $c_2(\odot^m E) = s_0 + s_1 + \dots + s_{\frac{m-1}{2}}$ . By simple algebra, we have

$$s_0 = \sum_{i=0}^{\frac{m-1}{2}} i(m-i)c_1(E)^2 + \sum_{i=0}^{\frac{m-1}{2}} (m-2i)^2 c_2(E).$$

Thus

$$\begin{aligned} c_2(\odot^m E) &= \sum_{i=0}^{\frac{m}{2}-1} (m-2i)^2 c_2(E) + \sum_{i=0}^{\frac{m}{2}-1} i(m-i)c_1(E)^2 + \\ &+ \sum_{j=1}^{\frac{m}{2}-1} \sum_{i=j}^{m-j} (m^2 - m(i+j) + 2ij)c_1(E)^2 \end{aligned}$$

if  $m$  is even and

$$\begin{aligned} c_2(\odot^m E) &= \sum_{i=0}^{\frac{m-1}{2}} (m-2i)^2 c_2(E) + \sum_{i=0}^{\frac{m-1}{2}} i(m-i)c_1(E)^2 + \\ &+ \sum_{j=1}^{\frac{m-1}{2}} \sum_{i=j}^{m-j} (m^2 - m(i+j) + 2ij)c_1(E)^2 \end{aligned}$$

if  $m$  is odd. The Lemma follows by simplifying the preceding formulas. QED

**Lemma 2.2** *Let  $E_i, i = 1, \dots, k$  be holomorphic vector bundles, of rank  $r_i$  respectively, over a non-singular complex surface  $X$  then*

$$\begin{aligned} (i) \quad c_1(\otimes_{i=1}^k E_i) &= \sum_{i=1}^k (r_1 \dots r_{i-1} r_{i+1} \dots r_k) c_1(E_i), \\ (ii) \quad c_2(\otimes_{i=1}^k E_i) &= \prod_{i=1}^k r_i \sum_{i=1}^k \left( \frac{c_2(E_i)}{r_i} - \frac{c_1^2(E_i)}{2r_i} \right) + \frac{\prod_{i=1}^k r_i^2}{2} \sum_{i=1}^k \left( \frac{c_1(E_i)}{r_i} \right)^2. \end{aligned}$$

*Proof.* Consider first the case  $k = 2$  then by expressing formally  $E_1 = L_1 \oplus \dots \oplus L_{r_1}, E_2 = F_1 \oplus \dots \oplus F_{r_2}$  as direct sums of line bundles we get

$$E_1 \otimes E_2 = \sum_{i=1}^{r_1} L_i \otimes (F_1 \oplus \dots \oplus F_{r_2})$$

hence the first Chern class is given by

$$c_1(E_1 \otimes E_2) = \sum_{i=1}^{r_1} (r_2 c_1(L_i) + c_1(E_2)) = \sum_{i=1}^{r-1} r_2 c_1(E_1) + r_1 c_2(E_2).$$

The case of general  $k$  is similar. For  $c_2(E_1 \otimes E_2)$  we have

$$\begin{aligned} c_2(E_1 \otimes E_2) &= c_2\left(\sum_{i=1}^{r_1} L_i \otimes E_2\right) \\ &= \sum_{i < j} c_1(L_i \otimes E_2) c_1(L_j \otimes E_2) + \sum_i c_2(L_i \otimes E_2). \end{aligned}$$

The formula of the Lemma follows from the above and the following formula,

$$c_l(L_i \otimes E_2) = \sum_{p=0}^l C_{l-p}^{r_2-p} c_1^{l-p}(L_p) c_i(E_2).$$

The calculation of the general case is achieved via induction. QED

The next formulas are consequences of the preceding lemmas:

**Corollary 2.3** *Let  $X$  be a non-singular complex surface  $X$  then*

$$c_1(\odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X) = \frac{i_1 + \dots + i_k}{2} (i_1 + 1) \dots (i_k + 1),$$

$$\begin{aligned} c_2(\odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X) &= \sum_{j=1}^k \frac{\prod_{l=1}^k (i_l + 1)}{i_j + 1} c_2(\odot^{i_j} T^* X) \\ &\quad + \sum_{j=1}^k \frac{\prod_{l \neq j} (i_l + 1) \{(\prod_{l \neq j} (i_l + 1)) - 1\}}{2} c_1^2(\odot^{i_j} T^* X) \\ &\quad + \prod_{1 \leq j_1 < j_2 \leq k} \frac{\prod_l (i_l + 1)^2}{(i_{j_1} + 1)(i_{j_2} + 1)} c_1(\odot^{i_{j_1}} T^* X) c_1(\odot^{i_{j_2}} T^* X). \end{aligned}$$

Let  $m$  be a positive integer and for each fixed positive integer  $k$  denote by  $q_k(m)$  be the number of solutions of the equation

$$i_1 + 2i_2 + \dots + ki_k = m.$$

A solution of the preceding equation shall be referred to as a weighted partition of  $m$  of length  $k$ . It is easy to see that

**Lemma 2.4** *With the notations above we have  $q_k(m) = p_1(m) + \dots + p_k(m)$ .*

With this we have the following asymptotic estimate:

**Theorem 2.5** *For  $m \rightarrow \infty$  the number  $q_k(m)$  is asymptotically given by:*

$$q_k(m) \sim \frac{m^{k-1}i^2)m^{j-1}}{(k-1)!(k-1)!}.$$

*Proof.* By Theorem 1.20, we have

$$p_j(m) \sim \frac{m^{j-1}}{(j-1)!j!}.$$

By Lemma 2.4,

$$q_k(m) = \sum_{j=1}^k p_j(m) \sim \sum_{j=1}^k \frac{m^{j-1}}{(j-1)!j!} \sim \frac{m^{k-1}i^2)m^{j-1}}{(k-1)!(k-1)!}.$$

QED

With the preceding results the computation of the Chern numbers for  $\mathcal{J}_k^m X$  can now be carried out by using the Theorem of Green on Griffiths. First we compute the Chern classes for the sheaves of each of the weighted partitions. Then the Chern numbers of  $\mathcal{J}_k^m X$  is computed from these by the following Lemma. To state the Lemma we denote by

$$\mathcal{I}_{km} = \{I = (i_1, \dots, i_k) \mid i_j \in \mathbf{N}, i_1 + 2i_2 + \dots + ki_k = m\}.$$

Moreover fixing an ordering of the set  $\mathcal{I}_{km}$  then

**Lemma 2.6** *Let  $X$  be a non-singular surface then*

$$\begin{aligned} c_1(\mathcal{J}_k^m X) &= \sum_{I \in \mathcal{I}_{km}} c_1(\mathcal{S}_I), \\ c_2(\mathcal{J}_k^m X) &= \sum_I c_2(\mathcal{S}_I) + \sum_{I < J, I, J \in \mathcal{I}_{km}} c_1(\mathcal{S}_I)c_1(\mathcal{S}_J) \end{aligned}$$

where  $\mathcal{S}_I = \odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X$ .

*Proof.* This is a consequence of Theorem 1.13:

$$\mathcal{J}_{k-1}^m X = \mathcal{F}_k^0 \subset \mathcal{F}_k^1 \subset \dots \subset \mathcal{F}_k^{[m/k]} = \mathcal{J}_k^m X$$

(where  $[m/k]$  is the greatest integer smaller than or equal to  $m/k$ ) such that

$$\mathcal{F}_k^i / \mathcal{F}_k^{i-1} \cong \mathcal{J}_{k-1}^{m-k} X \otimes (\odot^i T^* X).$$

From the exact sequence

$$0 \rightarrow \mathcal{F}_k^{[m/k]-1} \rightarrow \mathcal{J}_k^m X \rightarrow \mathcal{J}_{k-1}^{m-k[m/k]} X \otimes (\odot^{[m/k]} T^* X) \rightarrow 0$$

we see that

$$c_1(\mathcal{J}_k^m X) = c_1(\mathcal{F}_k^{[m/k]-1}) + c_1(\mathcal{J}_{k-1}^{m-k[m/k]} X \otimes (\odot^{[m/k]} T^* X)),$$

$$\begin{aligned} c_2(\mathcal{J}_k^m X) &= c_1(\mathcal{F}_k^{[m/k]-1}) c_1(\mathcal{J}_{k-1}^{m-k[m/k]} X \otimes (\odot^{[m/k]} T^* X)) \\ &\quad + c_2(\mathcal{F}_k^{[m/k]-1}) + c_2(\mathcal{J}_{k-1}^{m-k[m/k]} X \otimes (\odot^{[m/k]} T^* X)). \end{aligned}$$

We then use filtrations of  $\mathcal{F}_k^{[m/k]-1}$  and of  $\mathcal{J}_{k-1}^{m-k[m/k]} X$  to compute the Chern classes. Eventually the Chern classes are expressed by the Chern classes of the bundles  $\mathcal{S}_I = \odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X$  for each  $I \in \mathcal{I}_{km}$ . QED

We shall compute the explicit numbers for the following cases (I)  $k = 2, 1 \leq m \leq 6$ , (II)  $k = 3, m = 6$  which will be needed later. We shall also compute (III)  $k = m \leq 5$  for comparison with the result of section 1. We shall write, for simplicity:

$$c_1 = c_1(T^* X), c_2 = c_2(T^* X).$$

(I<sub>22</sub>)  $k = 2, m = 2$

There are two weighted partitions  $P_1 = (i_1 = 2, i_2 = 0)$  and  $P_2 = (i_1 = 0, i_2 = 1)$  corresponding to the two solutions of  $i_1 + 2i_2 = 2$ . The corresponding sheaves are  $\mathcal{S}_1 = \odot^2 T^* X, \mathcal{S}_2 = T^* X$ . Denote by  $\Delta(\mathcal{S}_i) = c_1(\mathcal{S}_i) - c_2(\mathcal{S}_i)$  and  $\mu(\mathcal{S}_i) = c_1(\mu(\mathcal{S}_i)) / \text{rank } \mu(\mathcal{S}_i)$ .

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
$(2, 0)$	$\odot^2 T^* X$	3	$3c_1$	$2c_1^2 + 4c_2$	$7c_1^2 - 4c_2$	1
$(0, 1)$	$T^* X$	2	$c_1$	$c_2$	$c_1^2 - 4c_2$	$1/2$

Thus  $c_1(\mathcal{J}_2^2 X) = 4c_1(T^* X)$ ,  $c_2(\mathcal{J}_2^2 X) = 5c_1^2(T^* X) + 5c_2(T^* X)$ , hence

$$\Delta(\mathcal{J}_2^2 X) = c_1^2(\mathcal{J}_2^2 X) - c_2(\mathcal{J}_2^2 X) = 11c_1^2(T^* X) - 5c_2(T^* X), \quad \mu(\mathcal{J}_2^2 X) = 4/5.$$

We remark that the formula given in [G-G] is  $c_1^2(\mathcal{J}_2^2 X) - c_2(\mathcal{J}_2^2 X) = 7c_1^2(T^* X) - 5c_2(T^* X)$ .

$(I_{23})$   $k = 2, m = 3$

There are two weighted partitions  $P_1 = (i_1 = 3, i_2 = 0)$  and  $P_2 = (i_1 = 1, i_2 = 1)$  corresponding to the two solutions of  $i_1 + 2i_2 = 3$ .

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
$(3, 0)$	$\odot^3 T^* X$	4	$6c_1$	$11c_1^2 + 10c_2$	$25c_1^2 - 10c_2$	$3/2$
$(1, 1)$	$T^* X \otimes T^* X$	4	$4c_1$	$6c_1^2 + 4c_2$	$10c_1^2 - 4c_2$	1

Thus  $c_1(\mathcal{J}_2^3 X) = 10c_1(T^* X)$ ,  $c_2(\mathcal{J}_2^3 X) = 41c_1^2(T^* X) + 14c_2(T^* X)$ , hence

$$\Delta(\mathcal{J}_2^3 X) = 59c_1^2(T^* X) - 14c_2(T^* X), \quad \mu(\mathcal{J}_2^3 X) = 5/4.$$

$(I_{24})$   $k = 2, m = 4$

There are 3 weighted partitions  $P_1 = (i_1 = 4, i_2 = 0)$ ,  $P_2 = (i_1 = 2, i_2 = 1)$  and  $P_3 = (i_1 = 0, i_2 = 2)$  corresponding to the 3 solutions of  $i_1 + 2i_2 = 4$ .

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
$(4, 0)$	$\odot^4 T^* X$	5	$10c_1$	$35c_1^2 + 20c_2$	$65c_1^2 - 20c_2$	2
$(2, 1)$	$\odot^2 T^* X \otimes T^* X$	6	$9c_1$	$34c_1^2 + 11c_2$	$57c_1^2 - 11c_2$	$3/2$
$(0, 2)$	$\odot^2 T^* X$	3	$3c_1$	$2c_1^2 + 4c_2$	$7c_1^2 - 4c_2$	1

Thus  $c_1(\mathcal{J}_2^4 X) = 22c_1(T^* X)$ ,  $c_2(\mathcal{J}_2^4 X) = 203c_1^2(T^* X) + 35c_2(T^* X)$ , hence

$$\Delta(\mathcal{J}_2^4 X) = 281c_1^2(T^* X) - 35c_2(T^* X), \quad \mu(\mathcal{J}_2^4 X) = 11/7.$$

( $I_{25}$ )  $k = 2, m = 5$

There are 3 weighted partitions  $P_1 = (i_1 = 5, i_2 = 0)$ ,  $P_2 = (i_1 = 3, i_2 = 1)$  and  $P_3 = (i_1 = 1, i_2 = 2)$  corresponding to the 3 solutions of  $i_1 + 2i_2 = 5$ .

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
(5, 0)	$\odot^5 T^* X$	6	$15c_1$	$85c_1^2 + 35c_2$	$120c_1^2 - 35c_2$	$5/2$
(3, 1)	$\odot^3 T^* X \otimes T^* X$	8	$16c_1$	$112c_1^2 + 24c_2$	$144c_1^2 - 24c_2$	2
(1, 2)	$T^* X \otimes \odot^2 T^* X$	6	$9c_1$	$34c_1^2 + 11c_2$	$47c_1^2 - 11c_2$	$3/2$

Thus  $c_1(\mathcal{J}_2^5 X) = 40c_1(T^* X)$ ,  $c_2(\mathcal{J}_2^5 X) = 750c_1^2(T^* X) + 70c_2(T^* X)$ , hence

$$\Delta(\mathcal{J}_2^5 X) = c_1^2(\mathcal{J}_2^5 X) - c_2(\mathcal{J}_2^5 X) = 850c_1^2(T^* X) - 70c_2(T^* X), \quad \mu(\mathcal{J}_2^5 X) = 2.$$

( $I_{26}$ )  $k = 2, m = 6$

There are 4 weighted partitions  $P_1 = (i_1 = 6, i_2 = 0)$ ,  $P_2 = (i_1 = 4, i_2 = 1)$ ,  $P_3 = (i_1 = 2, i_2 = 1)$  and  $P_4 = (i_1 = 0, i_2 = 3)$  corresponding to the 3 solutions of  $i_1 + 2i_2 = 6$ .

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
(6, 0)	$\odot^6 T^* X$	7	$21c_1$	$175c_1^2 + 56c_2$	$226c_1^2 - 56c_2$	3
(4, 1)	$\odot^4 T^* X \otimes T^* X$	10	$25c_1$	$330c_1^2 + 45c_2$	$295c_1^2 - 45c_2$	$5/2$
(2, 2)	$\odot^2 T^* X \odot^2 T^* X$	9	$18c_1$	$147c_1^2 + 24c_2$	$177c_1^2 - 24c_2$	2
(0, 3)	$\odot^3 T^* X$	4	$6c_1$	$11c_1^2 + 10c_2$	$25c_1^2 - 10c_2$	$3/2$

Thus  $c_1(\mathcal{J}_2^6 X) = 70c_1(T^* X)$ ,  $c_2(\mathcal{J}_2^6 X) = 662c_1^2(T^* X) + 135c_2(T^* X)$ , hence

$$\Delta(\mathcal{J}_2^6 X) = 4238c_1^2(T^* X) - 135c_2(T^* X), \quad \mu(\mathcal{J}_2^6 X) = 7/3.$$

( $II_{36}$ )  $k = 3, m = 6$

There are 7 weighted partitions  $P_1 = (i_1 = 6, i_2 = 0, i_3 = 0)$ ,  $P_2 = (i_1 = 4, i_2 = 1, i_3 = 0)$ ,  $P_3 = (i_1 = 3, i_2 = 0, i_3 = 1)$ ,  $P_4 = (i_1 = 2, i_2 = 2, i_3 = 0)$ ,  $P_5 = (i_1 = 1, i_2 = 1, i_3 = 1)$ ,  $P_6 = (i_1 = 0, i_2 = 3, i_3 = 0)$  and  $P_7 = (i_1 = 0, i_2 = 0, i_3 = 2)$  corresponding to the 7 solutions of  $i_1 + 2i_2 + 3i_3 = 6$ .

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
(6, 0, 0)	$\odot^6 T^* X$	7	$21c_1$	$175c_1^2 + 56c_2$	$260c_1^2 - 56c_2$	3
(4, 1, 0)	$\odot^4 T^* X \otimes T^* X$	10	$25c_1$	$330c_1^2 + 45c_2$	$295c_1^2 - 45c_2$	5/2
(3, 0, 1)	$\odot^3 T^* X \otimes T^* X$	8	$16c_1$	$112c_1^2 + 24c_2$	$144c_1^2 - 24c_2$	2
(2, 2, 0)	$\odot^2 T^* X \otimes \odot^2 T^* X$	9	$18c_1$	$147c_1^2 + 24c_2$	$177c_1^2 - 24c_2$	2
(1, 1, 1)	$T^* X \otimes T^* X \otimes T^* X$	8	$12c_1$	$66c_1^2 + 12c_2$	$78c_1^2 - 12c_2$	3/2
(0, 3, 0)	$\odot^3 T^* X$	4	$6c_1$	$11c_1^2 + 10c_2$	$25c_1^2 - 10c_2$	3/2
(0, 0, 2)	$\odot^2 T^* X$	3	$3c_1$	$2c_1^2 + 4c_2$	$7c_1^2 - 4c_2$	1

us  $c_1(\mathcal{J}_3^6 X) = 101c_1(T^* X)$ ,  $c_2(\mathcal{J}_3^6 X) = 5026c_1^2(T^* X) + 175c_2(T^* X)$  and

$$\Delta(\mathcal{J}_3^6 X) = 5175c_1^2(T^* X) - 175c_2(T^* X), \mu(\mathcal{J}_3^6 X) = 101/49.$$

(III<sub>33</sub>)  $k = m = 3$

In this case there are 3 weighted partitions:  $P_1 = (3, 0, 0)$ ,  $P_2 = (1, 1, 0)$  and  $P_3 = (0, 0, 3)$ . The tabulation is given by

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
(3, 0, 0)	$\odot^3 T^* X$	4	$6c_1$	$11c_1^2 + 10c_2$	$25c_1^2 - 10c_2$	3/2
(1, 1, 0)	$T^* X \otimes T^* X$	4	$4c_1$	$6c_1^2 + 4c_2$	$10c_1^2 - 4c_2$	1
(0, 0, 1)	$T^* X$	2	$c_1$	$c_2$	$c_1^2 - c_2$	1

Thus we have

$$c_1(\mathcal{J}_3^3 X) = 11c_1^2(T^* X), c_2(\mathcal{J}_3^3 X) = 51c_1^2(T^* X) + 15c_2(T^* X)$$

and so  $\mu(\mathcal{J}_3^3 X) = 11/10$  and

$$c_1^2(\mathcal{J}_3^3 X) - c_2(\mathcal{J}_3^3 X) = 70c_1^2(T^* X) - 15c_2(T^* X).$$

The formula given in [G-G] is  $c_1^2(\mathcal{J}_3^3 X) - c_2(\mathcal{J}_3^3 X) = 85c_1^2(T^* X) - 49c_2(T^* X)$ .

(III<sub>44</sub>)  $k = m = 4$

In this case there are 5 weighted partitions:  $P_1 = (4, 0, 0, 0)$ ,  $P_2 = (2, 1, 0, 0)$ ,  $P_3 = (1, 0, 1, 0)$ ,  $P_4 = (0, 2, 0, 0)$  and  $P_5 = (0, 0, 0, 1)$ . The tabulation is given by

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
(4, 0, 0, 0)	$\odot^4 T^* X$	5	$10c_1$	$35c_1^2 + 20c_2$	$65c_1^2 - 20c_2$	2
(2, 1, 0, 0)	$\odot^2 T^* X \otimes T^* X$	6	$9c_1$	$34c_1^2 + 11c_2$	$47c_1^2 - 11c_2$	3/2
(1, 0, 1, 0)	$T^* X \otimes T^* X$	4	$4c_1$	$6c_1^2 + 4c_2$	$10c_1^2 - 4c_2$	1
(0, 2, 0, 0)	$\odot^2 T^* X$	3	$3c_1$	$2c_1^2 + 4c_2$	$7c_1^2 - 4c_2$	1
(0, 0, 0, 1)	$T^* X$	2	$c_1$	$c_2$	$c_1^2 - c_2$	1/2

Thus we have

$$c_1(\mathcal{J}_4^4 X) = 27c_1^2(T^* X), c_2(\mathcal{J}_4^4 X) = 338c_1^2(T^* X) + 40c_2(T^* X)$$

and so  $\mu(\mathcal{J}_4^4 X) = 27/20$  and

$$c_1^2(\mathcal{J}_4^4 X) - c_2(\mathcal{J}_4^4 X) = 391c_1^2(T^* X) - 40c_2(T^* X).$$

(III<sub>55</sub>)  $k = m = 5$

In this case there are 3 weighted partitions:  $P_1 = (5, 0, 0, 0, 9)$ ,  $P_2 = (3, 1, 0, 0, 0)$ ,  $P_3 = (2, 0, 1, 0, 0)$ ,  $P_4 = (1, 2, 0, 0, 0)$ ,  $P_5 = (1, 0, 0, 1, 0)$ ,  $P_6 = (1, 2, 0, 0, 0)$  and  $P_7 = (1, 2, 0, 0, 0)$ . The tabulation is given by

$P_i$	$\mathcal{S}_i$	rank	$c_1(\mathcal{S}_i)$	$c_2(\mathcal{S}_i)$	$\Delta(\mathcal{S}_i)$	$\mu(\mathcal{S}_i)$
(5, 0, 0, 0, 0)	$\odot^5 T^* X$	6	$15c_1$	$85c_1^2 + 35c_2$	$140c_1^2 - 35c_2$	5/2
(3, 1, 0, 0, 0)	$\odot^3 T^* X \otimes T^* X$	8	$16c_1$	$112c_1^2 + 24c_2$	$144c_1^2 - 24c_2$	2
(2, 0, 1, 0, 0)	$\odot^2 T^* X \otimes T^* X$	6	$9c_1$	$34c_1^2 + 11c_2$	$47c_1^2 - 11c_2$	3/2
(1, 2, 0, 0, 0)	$T^* X \otimes \odot^2 T^* X$	6	$9c_1$	$34c_1^2 + 11c_2$	$47c_1^2 - 11c_2$	3/2
(1, 0, 0, 1, 0)	$T^* X \otimes T^* X$	4	$4c_1$	$6c_1^2 + 4c_2$	$10c_1^2 - 4c_2$	1
(0, 1, 1, 0, 0)	$T^* X \otimes T^* X$	4	$4c_1$	$6c_1^2 + 4c_2$	$10c_1^2 - 4c_2$	1
(0, 0, 0, 0, 1)	$T^* X$	2	$c_1$	$c_2$	$c_1^2 - c_2$	1

Thus we have

$$c_1(\mathcal{J}_5^5 X) = 58c_1^2(T^* X), c_2(\mathcal{J}_5^5 X) = 1622c_1^2(T^* X) + 90c_2(T^* X)$$

and so  $\mu(\mathcal{J}_5^5 X) = 29/18$  and

$$c_1^2(\mathcal{J}_5^5 X) - c_2(\mathcal{J}_5^5 X) = 1742c_1^2(T^* X) - 90c_2(T^* X).$$



We remark that the inequality (1.21) in [G-G] is incorrect (for example set  $k = 2$  or  $k = 3$  and compare these to the formulas obtained above; indeed for  $k = 2$  (see (1.21) in [G-G]) reduces to  $c_1^2(X) - c_2(X) > 0$ ).

### § 3 Weighted Projective Spaces and Projectivized Jet Bundles

For a vector bundle, e.g., the  $k$ -jet bundle  $T^k X$ , a standard approach of studying the bundle is to projectivize it and then study the line bundles over the projectivization. We are going to do the same for the  $\mathbf{C}^*$ -bundle  $J^k X$  using the well-known results in the former case as a guide. The fiber of the projectivized bundle are certain types of weighted projective space. Thus we shall first recall some basic facts about weighted projective spaces. For more detailed discussions and further references the readers are referred to the articles [B-R], [Do] and the monograph [Di].

Let  $Q = (q_0, q_1, \dots, q_r)$  ( $r \geq 1$ ) be an  $(r + 1)$ -tuple of positive integers. The tuple  $Q$  is said to be *reduced* if the greatest common divisor (gcd) of  $(q_0, q_1, \dots, q_r)$  is 1. In general if the gcd is  $d$  the tuple

$$Q_{\text{red}} = Q/d = (q_0/d, \dots, q_r/d)$$

is called the reduction of  $Q$ . Let  $d_0 = \gcd(q_1, \dots, q_r)$ ,  $d_r = \gcd(q_0, \dots, q_{r-1})$  and

$$d_i = \gcd(q_0, q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_r), \quad 1 \leq i \leq r - 1.$$

Let  $a_0 = \text{lcm}(d_1, \dots, d_r)$ ,  $a_r = \text{lcm}(d_0, \dots, d_{r-1})$  and

$$a_i = \text{lcm}(d_0, d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_r), \quad 1 \leq i \leq r - 1$$

where  $\text{lcm}$  means least common multiple. Define the normalization of  $Q$  by

$$Q_{\text{norm}} = (q_0/a_0, \dots, q_r/a_r).$$

A tuple  $Q$  is said to be *normalized* if  $Q = Q_{\text{norm}}$ .

Let  $(\mathbf{C}^{r+1}, Q)$  be the  $(r + 1)$ -dimensional complex vector space such that the variable  $z_i$  is assigned the weight (or degree)  $q_i$ . A  $\mathbf{C}^*$ -action is defined on  $(\mathbf{C}^{r+1}, Q)$  by:

$$\lambda.(z_0, \dots, z_r) = (\lambda^{q_0} z_0, \dots, \lambda^{q_r} z_r), \quad \lambda \in \mathbf{C}^*. \quad (24)$$

The quotient space,  $\mathbf{P}(Q) = (\mathbf{C}^{r+1}, Q)/\mathbf{C}^*$ , is called the weighted projective space of type  $Q$ . The equivalence class of an element  $(z_0, \dots, z_r)$  is denoted by  $[z_0, \dots, z_r]_Q$ . For  $Q = (1, \dots, 1) = \mathbf{1}$ ,  $\mathbf{P}(Q) = \mathbf{P}^r$  is the usual complex projective space of dimension  $r$  and an element of  $\mathbf{P}^r$  is denoted simply by  $[z_0, \dots, z_r]$ . Indeed for the special case  $r = 1$  it can be shown that, for any tuple  $(q_0, q_1)$ ,  $\mathbf{P}(q_0, q_1) \cong \mathbf{P}^1$ . This is not so if  $r \geq 2$ , however, we do have:

**Theorem 3.1** *Let  $Q = (q_0, \dots, q_r)$  be an  $(r + 1)$ -tuple of positive integers then*

$$\mathbf{P}(Q) \cong \mathbf{P}(Q_{\text{red}}) \cong \mathbf{P}(Q_{\text{norm}}).$$

**Example 3.2** It is clear that a normalized tuple is reduced. The converse is not true in general. Let  $Q = (4, 6, 12)$  then  $Q_{\text{red}} = (2, 3, 6)$  is reduced but is not normalized. In fact  $Q_{\text{norm}} = (Q_{\text{red}})_{\text{norm}} = (1, 1, 6)$ . The tuple  $(6, 10, 15)$  is reduced but is not normalized, in fact its normalization is  $(1, 1, 1)$  hence  $\mathbf{P}(6, 10, 15) \cong \mathbf{P}^2$ .

Define a map  $\rho_Q : (\mathbf{C}^{r+1}, \mathbf{1}) \rightarrow (\mathbf{C}^{r+1}, Q)$  by

$$\rho_Q(z_0, \dots, z_r) = (z_0^{q_0}, \dots, z_r^{q_r}). \quad (25)$$

It is easily seen that  $\mu_Q$  is compatible with the respective  $\mathbf{C}^*$ -actions and hence descends to a well-defined morphism:

$$\bar{\rho}_Q : \mathbf{P}^r \rightarrow \mathbf{P}(Q), \quad \bar{\rho}_Q([z_0, \dots, z_r]) = [z_0^{q_0}, \dots, z_r^{q_r}]_Q. \quad (26)$$

The weighted projective space can also be described as follows. Denote by  $\Theta_{q_i}$  the group of  $q_i$ -th roots of unity. Then the group  $\Theta_Q = \oplus_{i=0}^r \Theta_{q_i}$  acts on  $\mathbf{P}^r$  by coordinate wise multiplication:

$$(\theta_0, \dots, \theta_r) \cdot [z_0, \dots, z_r] = [\theta_0 z_0, \dots, \theta_r z_r], \quad \theta_i \in \Theta_{q_i}$$

and it is easily verified that  $\mathbf{P}(Q) = \mathbf{P}^r / \Theta_Q$ .

**Theorem 3.3** *The weighted projective space  $\mathbf{P}(Q)$  is isomorphic to the quotient  $\mathbf{P}^r / \Theta_Q$ . In particular,  $\mathbf{P}(Q)$  is irreducible and normal (the singularities are cyclic quotients and hence rational).*

Denote by  $S_Q(m)$  the space of homogeneous polynomials of degree  $m > 0$  in the variables  $z_i$  (assigned with the degree  $q_i$ ). In other words, a polynomial  $P$  is in  $S(Q)(m)$  if

$$P(\lambda \cdot (z_0, \dots, z_r)) = \lambda^m P(z_0, \dots, z_r).$$

We may express such a polynomial explicitly:

$$P = \sum_{(i_0, \dots, i_r) \in \mathcal{I}_{Q,m}} a_{i_0 \dots i_r} z_0^{i_0} \dots z_r^{i_r} \quad (27)$$

where the index set  $\mathcal{I}_{Q,m}$  is defined by:

$$\mathcal{I}_{Q,m} = \{(i_0, \dots, i_r) \mid \sum_{j=0}^r q_j i_j = m\}.$$

The sheaf  $\mathcal{O}_{\mathbf{P}(Q)}(m)$  is the sheaf over  $\mathbf{P}(Q)$  whose global regular sections are precisely the elements of  $S_Q(m)$ :

$$H^0(\mathbf{P}(Q), \mathcal{O}_{\mathbf{P}(Q)}(m)) = S_Q(m). \quad (28)$$

For negative integer  $-m, m > 0$  the sheaf  $\mathcal{O}_{\mathbf{P}(Q)}(-m)$  is defined to be the dual of  $\mathcal{O}_{\mathbf{P}(Q)}(m)$ .

**Theorem 3.4** (i) For any  $m \in \mathbf{Z}$ ,  $\mathcal{O}_{\mathbf{P}(Q)}(m)$  is a reflexive coherent sheaf. (ii) The sheaf  $\mathcal{O}_{\mathbf{P}(Q)}(m)$  is locally free if  $m$  is divisible by each  $q_i$  (hence by the least common multiple). (iii) Let  $m_Q$  be the least common multiple of  $\{q_0, \dots, q_r\}$  then  $\mathcal{O}_{\mathbf{P}(Q)}(m_Q)$  is ample. (iv) There exists an integer  $n_0$  depending only on  $Q$  such that  $\mathcal{O}_{\mathbf{P}(Q)}(nm_Q)$  is very ample for all  $n \geq n_0$ . (v) For any  $\alpha, \beta \in \mathbf{Z}$  we have  $\mathcal{O}_{\mathbf{P}(Q)}(\alpha m_Q) \otimes \mathcal{O}_{\mathbf{P}(Q)}(\beta) \cong \mathcal{O}_{\mathbf{P}(Q)}(\alpha m_Q + \beta)$ .

For any subset  $J \subset \{0, 1, \dots, r\}$  denote by  $m_J$  the least common multiple of  $\{q_j, j \in J\}$  and define

$$m(Q) = -|Q| + \frac{1}{r} \sum_{\nu=2}^{r+1} \frac{\sum_{\#J=\nu} m_J}{C_{\nu-2}^{r-1}}$$

where  $C_a^b$  is the usual binomial coefficient and  $|Q| = q_0 + \dots + q_r$ . It is known that assertion (iv) holds if  $n > m(Q)$ . In general the line sheaf  $\mathcal{O}_{\mathbf{P}(Q)}(m)$  is not invertible if  $m$  is not an integer multiple of  $m_Q$ . It can be shown that for  $Q = (1, 1, 2)$  the sheaf  $\mathcal{O}_{\mathbf{P}(Q)}(1)$  is not invertible and hence, neither is

$\mathcal{O}_{\mathbf{P}(Q)}(1) \otimes \mathcal{O}_{\mathbf{P}(Q)}(1)$ . This also shows that  $\mathcal{O}_{\mathbf{P}(Q)}(1) \otimes \mathcal{O}_{\mathbf{P}(Q)}(1) \not\cong \mathcal{O}_{\mathbf{P}(Q)}(2)$  as  $\mathcal{O}_{\mathbf{P}(Q)}(2)$  is invertible by part (ii) of the preceding Theorem.

**Theorem 3.5** *Let  $Q$  be a  $(r+1)$ -tuple of positive integers then*

- (i)  $H^i(\mathbf{P}(Q), \mathcal{O}_{\mathbf{P}(Q)}(p)) = \{0\}$ ,  $p \in \mathbf{Z}$  if  $i \neq 0, r$ ;
- (ii)  $H^0(\mathbf{P}(Q), \mathcal{O}_{\mathbf{P}(Q)}(p)) = S_Q(p)$   $p \in \mathbf{Z}$ ;
- (iii)  $H^r(\mathbf{P}(Q), \mathcal{O}_{\mathbf{P}(Q)}(p)) \cong S(Q)(-p - |Q|)$ ,  $p \in \mathbf{Z}$

where  $|Q| = q_0 + \dots + q_r$ .

Denote by  $\text{Pic}(\mathbf{P}(Q))$  and  $\text{Cl}(\mathbf{P}(Q))$  the Picard group and respectively the divisor class group.

**Theorem 3.6** *Let  $Q = Q_{\text{norm}}$  be a normalized  $(r+1)$ -tuple of positive integers then (i)  $\text{Pic}(\mathbf{P}(Q)) \cong \mathbf{Z}$  is generated by  $[\mathcal{O}_{\mathbf{P}(Q)}(m_Q)]$ ; (ii)  $\text{Cl}(\mathbf{P}(Q)) \cong \mathbf{Z}$  is generated by  $[\mathcal{O}_{\mathbf{P}(Q)}(1)]$ .*

Let  $Q$  be a  $(r+1)$ -tuple of positive integers define for  $k = 1, \dots, r$ :

$$l_{Q,k} = \text{lcm} \left\{ \frac{q_{i_0} \dots q_{i_k}}{\text{gcd}(q_0, \dots, q_{i_k})} \mid 0 \leq i_0 < \dots < i_k \leq r \right\}.$$

**Theorem 3.7** *Let  $Q$  be a  $(r+1)$ -tuple of positive integers then*

$$H^i(\mathbf{P}(Q); \mathbf{Z}) \cong \begin{cases} \mathbf{Z}, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Moreover, let  $\bar{\rho}_Q : \mathbf{P}^r \rightarrow \mathbf{P}(Q)$  be the quotient map as defined by (28) then the following diagram commutes,

$$\begin{array}{ccc} H^{2k}(\mathbf{P}(Q); \mathbf{Z}) & \xrightarrow{\bar{\rho}_Q^*} & H^{2k}(\mathbf{P}^r; \mathbf{Z}) \\ \cong \downarrow & & \cong \downarrow \\ \mathbf{Z} & \xrightarrow{l_{Q,k}} & \mathbf{Z} \end{array}$$

where the lower map is the multiplication by the number  $l_{Q,k}$ .

Note that the number  $l_{Q,r}$  is precisely the number of preimages of a point in  $\mathbf{P}(Q)$  under the quotient map  $\bar{\rho}_Q$ . The proof of the preceding Theorem

for  $k = r$  is quite easy. for the general case we refer the readers to [Ka]. We shall only be concerned with the case where  $n, k \geq 1$  are positive integers and

$$Q = (\underbrace{(1, \dots, 1)}_n, \underbrace{(2, \dots, 2)}_n, \dots, \underbrace{(k, \dots, k)}_n).$$

In this case we shall write  $\mathbf{P}_{n,k}$  for  $\mathbf{P}(Q)$ . Note that  $r = \dim \mathbf{P}_{n,k} = nk - 1$ . In this case the least common multiple of  $Q$  is  $m_Q = k!$  and  $l_{Qr} = (k!)^n$ .

Let  $\pi : (\mathcal{E}, h) \rightarrow X$  be a holomorphic hermitian vector bundle over a compact Kähler manifold  $X$ . Denote by  $\mathcal{L}(\mathcal{E})$  be the "hyperplane bundle" defined over the projectivized bundle  $\mathbf{P}(\mathcal{E})$ . It is defined as follows:

$$\begin{array}{ccc} \pi^* \mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow pr & & \downarrow p \\ \mathbf{P}(\mathcal{E}) & \xrightarrow{\pi} & X \end{array}$$

the tautological sub-sheaf is defined by:

$$\{((x, [\xi]), \eta) \in \pi^* \mathcal{E} \mid (x, [\xi]) \in \mathbf{P}(\mathcal{E}), p([\xi]) = x, [\eta] = [\xi]\}$$

and  $\mathcal{L}_k$  is defined to be the dual of the tautological line bundle. In other words, since the fiber  $\mathbf{P}(\mathcal{E})$  over a point  $x \in X$  is a projective space, the restriction of  $\mathcal{L}_k X$  to  $\mathbf{P}(\mathcal{E}_x)$  is the hyperplane line bundle  $\mathcal{O}_{\mathbf{P}^{r-1}}(1)$  (here  $r = \text{rank } \mathcal{E}$ ). We shall often use the notation  $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$  for  $\mathcal{L}_k$  and the tensor product  $\mathcal{L}_k^m$  by  $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(m)$  for any integer  $m \in \mathbf{Z}$ . The following is a classical Theorem of Grothendieck:

**Theorem 3.8** *Let  $\mathcal{E}$  be a holomorphic vector bundle over a complex manifold  $X$  then for any  $m, j \geq 0$ , the  $j$ -th direct image sheaf of the  $m$ -fold tensor product of  $\mathcal{L}(m)$  is isomorphic to the  $m$  fold symmetric product of  $E$ , i.e.,  $R_*^j \mathcal{L}^m(\mathcal{E}) \cong \odot^m \mathcal{E}$  and*

$$H^j(X, \odot^m \mathcal{E} \otimes \mathcal{S}) \cong H^q(X, \mathcal{L}^m(\mathcal{E}) \otimes p^* \mathcal{S})$$

where  $\mathcal{S}$  is any sheaf on  $X$ .

Let  $\pi : J^k X \rightarrow X$  be the (restricted)  $k$ -jet bundle of a complex manifold  $X$ . Denote by  $\mathcal{L}_k$  the "hyperplane sheaf" defined over the projectivized

$k$ -jet bundle  $\mathbf{P}(J^k X)$ . It is defined as follows. Consider the commutative diagram:

$$\begin{array}{ccc} \pi^* J^k X & \longrightarrow & J^k X \\ \downarrow pr & & \downarrow p \\ \mathbf{P}(J^k X) & \xrightarrow{\pi} & X \end{array}$$

the tautological sub-sheaf is defined by:

$$\{((x, [\xi]), \eta) \in \pi^* J^k X \mid (x, [\xi]) \in \mathbf{P}(J^k X), p([\xi]) = x, [\eta] = [\xi]\}$$

and  $\mathcal{L}_k$  is defined to be the dual the tautological line sheaf. In other words, since the fiber  $\mathbf{P}(J_x^k X)$  over a point  $x \in X$  is a weighted projective space of type  $Q = ((1, \dots, 1); \dots; (k, \dots, k))$  the restriction of  $\mathcal{L}_k X$  to  $\mathbf{P}(J_x^k X)$  is the line sheaf  $\mathcal{O}_{\mathbf{P}(Q)}(1)$  as defined in the preceding section. We shall use the notation  $\mathcal{O}_{\mathbf{P}(J^k X)}(1)$  for  $\mathcal{L}_k$ . More generally for any integer  $m$ ,  $\mathcal{O}_{\mathbf{P}(J^k X)}(m)$  is the sheaf on  $\mathbf{P}(J^k X)$  which restricts to the bundle  $\mathcal{O}_{\mathbf{P}(Q)}(m)$  along each fiber of the projection map  $p : \mathbf{P}(J^k X) \rightarrow X$ . The proof of the preceding Theorem relies on the classical Vanishing Theorem of cohomologies on projective spaces. The analogue of this for weighted projective spaces is provided by Theorem 3.3 and hence we have (see [G-G] and [K-O]):

**Theorem 3.9** *Let  $X$  be a complex manifold and  $\mathcal{S}$  be a sheaf over  $X$  then for any  $m, j \geq 0$  we have  $R_*^j \mathcal{O}_{\mathbf{P}(J^k X)}(m) \cong \mathcal{J}_k^m X$  and*

$$H^j(X, \mathcal{J}_k^m X \otimes \mathcal{S}) \cong H^j(\mathbf{P}(J^k X), \mathcal{O}_{\mathbf{P}(J^k X)}(m) \otimes p^* \mathcal{S}).$$

The following is a consequence of Theorem 3.4:

**Theorem 3.10** *Let  $X$  be a complex manifold then*

- (i) *for any  $m \in \mathbf{Z}$ ,  $\mathcal{O}_{\mathbf{P}(J^k X)}(m)$  is a reflexive coherent sheaf;*
- (ii) *the sheaf  $\mathcal{O}_{\mathbf{P}(J^k X)}(m)$  is locally free if  $m$  is divisible by each  $q_i$  (hence by the least common multiple  $k!$ );*
- (iii) *for any  $\alpha, \beta \in \mathbf{Z}$ ,  $\mathcal{O}_{\mathbf{P}(J^k X)}(k! \alpha) \otimes \mathcal{O}_{\mathbf{P}(J^k X)}(\beta) \cong \mathcal{O}_{\mathbf{P}(J^k X)}(k! \alpha + \beta)$ .*

Due to the fact that  $\mathcal{O}_{\mathbf{P}(J^k X)}(1)$  is not locally free and that, in general,  $\mathcal{O}_{\mathbf{P}(J^k X)}(a) \otimes \mathcal{O}_{\mathbf{P}(J^k X)}(a) \cong \mathcal{O}_{\mathbf{P}(J^k X)}(a + b)$  some of the proof of the results that are valid on projectivized vector bundle are not valid even though

modifications of the results can be obtained via alternative proofs. We establish some of the results (the counterparts in the case of projectivized vector bundle are well-known) that will be essential in the next section.

**Lemma 3.11** *Let  $X$  be a complex manifold of dimension  $n$  and let  $p : \mathbf{P}(J^k X) \rightarrow X$  be the projection map. Then the natural morphism:*

$$\phi : p^* p_* \mathcal{O}_{\mathbf{P}(J^k X)}(k!) \rightarrow \mathcal{O}_{\mathbf{P}(J^k X)}(k!)$$

*is surjective and*

$$\sum_{i=0}^n (-1)^i c_1^{r-i}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_i(\mathcal{J}_k^{k!} X) = 0$$

*where  $\mathcal{F}$  is the kernel of  $\phi$  and  $r = nk - 1$  is the fiber dimension of  $p$ .*

*Proof.* For simplicity we write  $\mathcal{O}(k!)$  for  $\mathcal{O}_{\mathbf{P}(J^k X)}(k!)$ . By definition the restriction of  $\mathcal{O}(k!)$  to a fiber of the projection map is  $\mathcal{O}_{\mathbf{P}(Q)}(k!)$  where

$$Q = ((1, \dots, 1); \dots; (k, \dots, k)).$$

Thus the least common multiple of the indices is  $k!$  and  $\mathcal{O}_{\mathbf{P}(Q)}(k!)$  is ample by Theorem 3.4. This implies that the map  $\phi$  is surjective (see for example [B-S]). By Theorem 3.9  $p^* p_* \mathcal{O}(k!) = p^* \mathcal{J}_k^{k!} X$  and so the sequence:

$$0 \rightarrow \mathcal{F} \rightarrow p^* \mathcal{J}_k^{k!} X \rightarrow \mathcal{O}(k!) \rightarrow 0$$

is exact. By Whitney's formula

$$\sum_{i=0}^r p^* c_i(\mathcal{J}_k^{k!} X) = (1 + c_1(\mathcal{O}(k!))) \cdot \sum_{i=0}^{r-1} c_i(\mathcal{F})$$

and hence

$$p^* c_i(\mathcal{J}_k^{k!} X) = c_1(\mathcal{O}(k!)) \cdot c_{i-1}(\mathcal{F}) + c_i(\mathcal{F})$$

for  $0 \leq i \leq r$  with  $c_{-1}(\mathcal{F}) = c_r(\mathcal{F}) = 0$  (as  $\text{rank } \mathcal{F} = r - 1$ ). We can eliminate the Chern classes of  $\mathcal{F}$  by first multiplying the preceding identity by  $c_1^{r-i}(\mathcal{O}(k!))$  and then take alternating sum; namely:

$$\begin{aligned} c_1^{r-1}(\mathcal{O}(k!)) \cdot p^* c_1(\mathcal{J}_k^{k!} X) &= c_1^r(\mathcal{O}(k!)) + c_1^{r-1}(\mathcal{O}(k!)) \cdot c_1(\mathcal{F}) \\ c_1^{r-2}(\mathcal{O}(k!)) \cdot p^* c_2(\mathcal{J}_k^{k!} X) &= c_1^{r-1}(\mathcal{O}(k!)) \cdot c_1(\mathcal{F}) + c_1^{r-2}(\mathcal{O}(k!)) \cdot c_2(\mathcal{F}) \\ c_1^{r-3}(\mathcal{O}(k!)) \cdot p^* c_3(\mathcal{J}_k^{k!} X) &= c_1^{r-1}(\mathcal{O}(k!)) \cdot c_2(\mathcal{F}) + c_1^{r-3}(\mathcal{O}(k!)) \cdot c_3(\mathcal{F}) \\ &\dots \\ c_1(\mathcal{O}(k!)) \cdot p^* c_{r-1}(\mathcal{J}_k^{k!} X) &= c_1^2(\mathcal{O}(k!)) \cdot c_{r-2}(\mathcal{F}) + c_1(\mathcal{O}(k!)) \cdot c_{r-1}(\mathcal{F}) \\ p^* c_r(\mathcal{J}_k^{k!} X) &= c_1(\mathcal{O}(k!)) \cdot c_{r-1}(\mathcal{F}) \end{aligned}$$

and multiply the  $i$ -identity above by  $(-1)^r$  and then taking the sum from  $i = 1$  to  $i = r$  yields

$$\sum_{i=1}^r (-1)^i c_1^{r-i}(\mathcal{O}(k!)) \cdot p^* c_i(\mathcal{J}_k^{k!} X) = -c_1^r(\mathcal{O}(k!)).$$

Moving the RHS to the LHS yields the identity of the Lemma. QED

Note that  $c_i(\mathcal{J}_k^{k!} X) = 0$  if  $i \geq n = \dim X$ .

**Lemma 3.12** *Let  $X$  be a compact complex manifold of complex dimension  $n$  then for any  $x \in X$ ,*

$$\int_{\mathbf{P}(J^k X)_x} c_1^{nk-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)|_{\mathbf{P}(J^k X)_x}) = (k!)^n$$

where  $\mathbf{P}(J^k X)_x$  is the fiber over  $x$ .

*Proof.* By definition the fiber,  $\mathbf{P}(J^k X)_x$ , over any point  $x \in X$  of the projection map of  $p : \mathbf{P}(J^k X) \rightarrow X$  is the weighted projective space

$$\mathbf{P}(\underbrace{(1, \dots, 1)}_n, \underbrace{(2, \dots, 2)}_n, \dots, \underbrace{(k, \dots, k)}_n)$$

of dimension  $nk - 1$ . By Theorem 3.7 the quotient map  $\bar{\rho}_Q : \mathbf{P}^{nk-1} \rightarrow \mathbf{P}(Q)$  is a finite morphism with sheet number  $l_{Q, nk-1} = (k!)^n$ . The generator of  $H^{2(nk-1)}(\mathbf{P}^{nk-1}; \mathbf{Z})$  is represented by the  $(nk - 1)$ -th power,  $\omega_{FS}^{nk-1}$ , of the the Fubini-Study metric  $\omega_{FS} = c_1(\mathcal{O}_{\mathbf{P}^{nk-1}}(1))$ . The Lemma follows readily as we have:

$$\int_{\mathbf{P}^{nk-1}} c_1^{nk-1}(\mathcal{O}_{\mathbf{P}^{nk-1}}(1)) = \int_{\mathbf{P}^{nk-1}} \omega_{FS}^{nk-1} = 1.$$

QED

**Theorem 3.13** *Let  $X$  be a compact complex manifold of complex dimension  $n$  then the following intersection formulas hold:*

$$c_1^{nk+j-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \dots \cdot p^* D_{n-j} = (k!)^n \Delta_j \cdot D_1 \cdot \dots \cdot D_{n-j}$$

for divisors  $D_1, \dots, D_{n-j}, j = 0, 1, \dots, n$  on  $X$ . The numbers  $\Delta_j$  is defined by setting  $\Delta_0 = 1, \Delta_1 = c_1(\mathcal{J}_k^{k!} X)$  and by the recursive relation:

$$\Delta_j = \sum_{i=1}^j (-1)^{i+1} \Delta_{j-i} \cdot c_i(\mathcal{J}_k^{k!} X), \quad j \geq 2.$$



*Proof.* Note that  $\dim \mathbf{P}(J^k X) = n(k+1) - 1$  and the fiber dimension,  $\dim \mathbf{P}(J^k X)_x = nk - 1$ . Thus, by fiber integration (Lemma 3.12),

$$c_1^{nk-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \cdots \cdot p^* D_n = (k!)^n \Delta_0 D_1 \cdot \cdots \cdot D_n$$

which is the case  $j = 0$ . By Lemma 3.11 with  $r = nk - 1$ ,

$$\sum_{i=0}^r (-1)^i c_1^{r-i}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_i(\mathcal{J}_k^{k!} X) = 0 \quad (29)$$

and, multiplying by  $p^* D_1 \cdot \cdots \cdot p^* D_{n-1}$ , we get

$$\begin{aligned} & c_1^{nk-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-1} \\ &= c_1^{nk-2}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_1(\mathcal{J}_k^{k!} X) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-1} \end{aligned}$$

as the rest of the terms vanish for dimension reason. Multiplying the above by  $c_1(\mathcal{O}_{\mathbf{P}(J^k X)}(k!))$  yields,

$$\begin{aligned} & c_1^{nk}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-1} \\ &= c_1^{nk-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_1(\mathcal{J}_k^{k!} X) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-1}. \end{aligned}$$

Fiber integration shows that the term on the right above equals

$$(k!)^n c_1(\mathcal{J}_k^{k!} X) \cdot D_1 \cdot \cdots \cdot D_{n-1} = (k!)^n \Delta_1 \cdot D_1 \cdot \cdots \cdot D_{n-1}.$$

This establish the Theorem for the case  $j = 1$ .

If we multiply (31) by  $p^* D_1 \cdot \cdots \cdot p^* D_{n-2}$  we are left with 3 terms (again for dimension reason):

$$\begin{aligned} & c_1^{nk-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-2} \\ &= c_1^{nk-2}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_1(\mathcal{J}_k^{k!} X) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-2} \\ &\quad - c_1^{nk-3}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_2(\mathcal{J}_k^{k!} X) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-2}. \end{aligned}$$

Now multiply the above by  $c_1^2(\mathcal{O}_{\mathbf{P}(J^k X)}(k!))$  then the LHS above is given by

$$c_1^{nk+1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-2}$$

while the first term on the RHS is given by (the case  $j = 1$ ):

$$\begin{aligned} & c_1^{nk}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_1(\mathcal{J}_k^{k!} X) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-2} \\ &= (k!)^n \Delta_1 \cdot c_1(\mathcal{J}_k^{k!} X) \cdot D_1 \cdot \cdots \cdot D_{n-2} \end{aligned}$$

and the second term on the right is given by (the case  $j = 0$ ):

$$\begin{aligned} & -c_1^{nk-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* c_2(\mathcal{J}_k^{k!} X) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-2} \\ &= -(k!)^n \Delta_0 c_2(\mathcal{J}_k^{k!} X) \cdot D_1 \cdot \cdots \cdot D_{n-2} \end{aligned}$$

Combining the above yields the case  $j = 2$ :

$$\begin{aligned} & c_1^{nk+1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-2} \\ &= (k!)^n \{ \Delta_1 \cdot c_1(\mathcal{J}_k^{k!} X) - \Delta_0 c_2(\mathcal{J}_k^{k!} X) \} \cdot D_1 \cdot \cdots \cdot D_{n-2} \\ &= (k!)^n \Delta_2 \cdot D_1 \cdot \cdots \cdot D_{n-2} \end{aligned}$$

as, by definition,

$$\Delta_2 = \Delta_1 \cdot c_1(\mathcal{J}_k^{k!} X) - \Delta_0 c_2(\mathcal{J}_k^{k!} X) = c_1^2(\mathcal{J}_k^{k!} X) - c_2(\mathcal{J}_k^{k!} X).$$

Thus the case  $j = 2$  is also established. Inductively, the procedure above yields:

$$\begin{aligned} & c_1^{n-k+j-2}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!)) \cdot p^* D_1 \cdot \cdots \cdot p^* D_{n-3} \\ &= \sum_{i=1}^j (-1)^{i-1} \Delta_{j-i} \cdot c_i(\mathcal{J}_k^{k!} X) \cdot D_1 \cdot \cdots \cdot D_{n-3} \\ &= (k!)^n \Delta_j \cdot D_1 \cdot \cdots \cdot D_{n-3}. \end{aligned}$$

QED

**Theorem 3.14** *Let  $X$  be a non-singular projective surface and assume that (i)  $c_1^2(\mathcal{J}_k^{k!} X) - c_2(\mathcal{J}_k^{k!} X) > 0$  and (ii)  $h^2(\mathcal{J}_k^{k!m}) = O(m^{(n+1)k-2})$ . Then  $\mathcal{J}_k^{k!} X$  is big.*

*Proof.* Let  $\mathbf{P}(J^k X)$  be the projectivized  $k$ -jet bundle. Then  $\dim \mathbf{P}(J^k X) = (n+1)k - 1$ . Riemann-Roch applied to the line bundle  $\mathcal{O}_{\mathbf{P}(J^k X)}(k!)$  yields

$$\chi(\mathcal{O}_{\mathbf{P}(J^k X)}(k!m)) = \frac{c_1^{(n+1)k-1}(\mathcal{O}_{\mathbf{P}(J^k X)}(k!))}{((n+1)k-1)!} m^{(n+1)k-1} + O(m^{(n+1)k-2}).$$

Theorem 3.13 and assumption (i) imply that there exists positive constant  $c > 0$  and positive integer  $m'_0$  such that

$$\begin{aligned}\chi(\mathcal{O}_{\mathbf{P}(J^k X)}(k!m)) &= \frac{c_1^2(\mathcal{J}_k^{k!} X) - c_2(\mathcal{J}_k^{k!} X)}{((n+1)k-1)!} m^{(n+1)k-1} + O(m^{(n+1)k-2}) \\ &\geq cm^{(n+1)k-1}\end{aligned}$$

for all  $m \geq m'_0$ . Theorem 3.8 implies that the same is true for  $\mathcal{J}_k^{k!m}$  i.e.  $\chi(\mathcal{J}_k^{k!m}) \geq cm^{r+1}$  and, a priori:

$$h^0(\mathcal{J}_k^{k!m}) + h^2(\mathcal{J}_k^{k!m}) > cm^{(n+1)k-1}$$

for all  $m \geq m'_0$ . The Theorem follows now from assumption (ii).

#### § 4 Surfaces of General Type

We recall first some well-known results on manifolds of general type.

**Theorem 4.1** *Let  $X$  be a minimal surface of general type then  $c_1^2(T^*X) > 0$ ,  $c_2(T^*X) > 0$  and  $c_1^2(T^*X) \geq 3c_2(T^*X)$ . Moreover, we have*

$$5c_1^2(T^*X) - c_2(T^*X) + 36 \geq 0, \quad \text{if } m \text{ is even,}$$

$$5c_1^2(T^*X) - c_2(T^*X) + 30 \geq 0, \quad \text{if } m \text{ is odd.}$$

Let  $L_0$  be a nef line bundle on a non-singular surface  $X$ . A coherent sheaf  $E$  over  $X$  is said to be semi-stable (resp. stable) with respect to  $L_0$  if  $c_1(E) \cdot c_1(L_0) \geq 0$  and if, for any coherent subsheaf  $0 \neq \mathcal{S}$  of  $E$ , we have:

$$\mu_{\mathcal{S}, L_0} \stackrel{\text{def}}{=} \frac{c_1(\mathcal{S}) \cdot c_1(L_0)}{\text{rank } \mathcal{S}} \leq \mu_{E, L_0} \stackrel{\text{def}}{=} \frac{c_1(E) \cdot c_1(L_0)}{\text{rank } E} \quad (30)$$

(resp.  $\mu_{\mathcal{S}, L_0} < \mu_{E, L_0}$ ).

If  $X$  is of general type then (see Maruyama [Ma])

**Theorem 4.2** *Let  $X$  be a surface of general type then  $\otimes^m T^*X, \odot^m T^*X$  are semi-stable with respect to the canonical bundle  $K_X = \det T^*X$ .*

Indeed we have:

**Theorem 4.3** *Let  $X$  be a minimal surface of general type. If  $D$  is a divisor in  $X$  such that  $H^0(X, E_k \otimes [-D]) \neq 0$  where  $E_k = (\odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X)$ ,  $i_1, \dots, i_k$  being positive integers then*

$$c_1(E_k) \cdot c_1(D) \leq \mu_{E_k} \leq \frac{m}{2} c_1^2(T^* X)$$

with  $m = i_1 + 2i_2 + \dots + ki_k$

*Proof.* This follows from the calculation of the Chern number  $c_1(E_k)$  in section 2. The computation there shows that

$$\mu_{E_k} \leq \frac{m}{2} c_1^2(T^* X)$$

with equality if and only if  $k = 1$ , i.e.,

$$\mu_{\odot^m T^* X} = \frac{c_1(\odot^m T^* X)}{\text{rank } \odot^m T^* X} \cdot c_1(T^* X) = \frac{\frac{m(m+1)}{2}}{m+1} c_1^2(T^* X) = \frac{m}{2} c_1^2(T^* X).$$

QED

Note that in general, if  $E$  is a vector bundle of rank  $r$  then

$$\text{rank } \odot^m E = \frac{(m+r-1)!}{(r-1)!m!}. \quad (31)$$

The Chern number  $c_1(\odot^m E)$  is given by (compare section 2)

$$c_1(\odot^m E) = \frac{1}{r!} \frac{(m+r-1)!}{(m-1)!} c_1(E). \quad (32)$$

This is done by induction on the rank of  $E$ . If  $\text{rank } E = 1$  then clearly we have  $c_1(\odot^m E) = mc_1(E)$ . If  $\text{rank } E = 2$  we may formally split the bundle  $E$  as direct sum of line bundles, i.e., we have an exact sequence:

$$0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0$$

so that there is a filtration

$$\odot^m E = F_0 \supset F_1 \supset \dots \supset F_{m+1} = 0$$

with  $F_i/F_{i+1} \cong L_1^i \otimes L_2^{m-i}$  and so, by Whitney's formula:

$$\begin{aligned}
c_1(\odot^m E) &= \sum_{i=0}^m c_1(F_i/F_{i+1}) \\
&= \sum_{i=0}^m c_1(L_1^i \otimes L_2^{m-i}) \\
&= \sum_{i=0}^m i c_1(L_1) + \sum_{i=0}^m (m-i) c_1(L_2) \\
&= \frac{m(m+1)}{2} (c_1(L_1) + c_1(L_2)) \\
&= \frac{m(m+1)}{2} c_1(E).
\end{aligned}$$

If  $\text{rank } E = 3$  then we split the bundle into a rank 2 bundle  $A$  and a line bundle  $L$ , i.e., we have an exact sequence:

$$0 \rightarrow F \rightarrow E \rightarrow L \rightarrow 0$$

so that there is a filtration

$$\odot^m E = F_0 \supset F_1 \supset \dots \supset F_{m+1} = 0$$

with  $F_i/F_{i+1} \cong \odot^i F \otimes L^{m-i}$  and the Chern number is given by:

$$\begin{aligned}
c_1(\odot^m E) &= \sum_{i=0}^m c_1(F_i/F_{i+1}) \\
&= \sum_{i=0}^m \{c_1(\odot^i F) + (\text{rank } \odot^i F)(m-i)c_1(L)\}
\end{aligned}$$

and by induction the RHS above is

$$\sum_{i=0}^m \frac{i(i+1)}{2} c_1(F) + \sum_{i=0}^m (i+1)(m-i) c_1(L)$$

hence

$$\begin{aligned}
c_1(\odot^m E) &= \sum_{i=0}^m \frac{i(i+1)}{2} c_1(F) + \sum_{i=0}^m (i+1)(m-i) c_1(L) \\
&= \frac{1}{6} \frac{(m+2)!}{(m-1)!} c_1(F) + m \sum_{i=0}^m (i+1) c_1(L) - \sum_{i=0}^m i(i+1) c_1(L)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6} \frac{(m+2)!}{(m-1)!} c_1(F) + \frac{m(m+1)}{2} c_1(L) - \frac{1}{3} \frac{(m+2)!}{(m-1)!} c_1(L) \\
&= \frac{1}{6} \frac{(m+2)!}{(m-1)!} c_1(E).
\end{aligned}$$

Ne that we have used the formula:

$$\sum_{i=0}^m i(i+1) = \frac{1}{3} \frac{(m+2)!}{(m-1)!}.$$

The general case is proved by induction using the formula:

$$\sum_{i=0}^m i(i+1)(i+2)\dots(i+k) = \frac{1}{k+2} \frac{(m+k+1)!}{(m-1)!}.$$

QED

Examples of surfaces of general type are provided by complete intersections in  $\mathbf{P}^n$ . A Smooth complete intersection of type  $(d_1, \dots, d_{n-r}), 1 \leq r \leq n-1$ , in  $\mathbf{P}^n$  is a smooth variety  $X$  of dimension  $r$  which is the transversal intersections of  $(n-r)$  hypersurfaces of degree  $d_1, \dots, d_{n-r}$  respectively. By the adjunction formula, the canonical bundle of a complete intersection  $X$  of type  $(d_1, \dots, d_{n-r})$  is given by the formula:

$$\mathcal{K}_X = \mathcal{O}_{P^n}(d_1 + \dots + d_{n-r} - (n+1))|_X = \mathcal{O}_X(d_1 + \dots + d_{n-r} - (n+1)).$$

The normal bundle  $\mathcal{N}_{X|Y}$  of a smooth hypersurface  $X$  in a smooth variety  $Y$  is given by

$$\mathcal{N}_{X|Y} = \mathcal{O}_Y(X)|_X = \mathcal{O}_X(X).$$

Thus for a hypersurface  $X_1$  of degree  $d_1$  in  $\mathbf{P}^n$ , the normal bundle

$$\mathcal{N}_{X_1|P^n} = \mathcal{O}_{P^1}(d_1)|_{X_1} = \mathcal{O}_{X_1}(d_1).$$

Inductively, for a smooth complete intersection  $X$  of type  $(d_1, \dots, d_{n-r})$  we get

$$\mathcal{N}_{X|P^n} = \oplus_{1 \leq i \leq n-r} \mathcal{O}_X(d_i).$$

To compute Chern classes of  $X$  we apply the Whitney formula to the exact sequence:

$$0 \longrightarrow TX \longrightarrow TP^n|_X \longrightarrow \mathcal{N}_{X|P^n} = \oplus_{1 \leq i \leq n-r} \mathcal{O}_X(d_i) \longrightarrow 0.$$

which yields the following formula for the total Chern classes:

$$c(TX) \cdot c(\mathcal{N}_{X|P^n}) = c(T\mathbf{P}^n|_X).$$

Operating symbolically, we get:

$$1 + c_1(TX) + \dots + c_r(TX) = (1 + \theta)^{n+1} / \prod_{1 \leq i \leq n-r} (1 + d_i \theta)$$

where

$$\theta^r = \prod_{1 \leq i \leq n-r} d_i.$$

Expanding formally the RHS above yields:

$$(1 + \theta)^{n+1} = 1 + C_1^{n+1} \theta + C_2^{n+1} \theta^2 + \dots + C_r^{n+1} \theta^r$$

$$(1 + d_i \theta)^{-1} = 1 - d_i \theta + (d_i \theta)^2 - \dots + (-1)^r (d_i \theta)^r, 1 \leq i \leq n - r.$$

Define polynomials  $p_q(0 \leq q \leq n - r)$  in  $d_1, \dots, d_{n-r}$  by  $p_0(d_1, \dots, d_{n-r}) = 1$ ,

$$p_q(d_1, \dots, d_{n-r}) = \sum_{1 \leq i_1 \leq \dots \leq i_q \leq n-r} d_{i_1} \dots d_{i_q} \quad 1 \leq q \leq n - r.$$

Then for  $0 \leq q \leq n - r$ , the Chern classes of  $X$  are given by:

$$c_q(TX) = \sum_{i=0}^q (-1)^i C_{q-i}^{n+1} p_i(d_1, \dots, d_{n-r}) \theta^q.$$

For hypersurface ( $r = n - 1$ ) the formulas above reduce to

$$c_q(TX) = \sum_{i=0}^q (-1)^i C_{q-i}^{n+1} d_i \theta^q, \quad 0 \leq q \leq n - 1.$$

For surfaces of complete intersections ( $r = 2$ ) and the formulas reduce to:

$$c_1(TX) = ((n+1) - \sum_{i=1}^{n-2} d_i) \theta,$$

$$c_2(TX) = \left\{ \frac{n(n+1)}{2} - (n+1) \sum_{i=1}^{n-2} d_i + \sum_{1 \leq i \leq j \leq n-2} d_i d_j \right\} \theta^2.$$

In particular, if  $d_1 = \dots = d_{n-2} = d$  then

$$c_1(X) = \{(n+1) - (n-2)d\} \theta,$$

$$c_2(X) = \left\{ \frac{n(n+1)}{2} - (n+1)(n-2)d + \frac{(n-1)(n-2)}{2}d^2 \right\} \theta^2.$$

For  $n = 3$  then

$$c_1(TX) = (4-d)\theta, \quad c_2(TX) = (6-4d+d^2)\theta^2;$$

equivalently, for the cotangent bundle, we have:

$$c_1(T^*X) = (d-4)\theta, \quad c_2(T^*X) = (6-4d+d^2)\theta^2.$$

We shall need a vanishing Theorem (see [G-G]) which is a consequence of a result of Bogomolov ([B1], [B2]):

**Theorem 4.4** *Let  $X$  be a minimal surface of general type and if the geometric genus  $p_g(X) > 0$  then*

$$H^2(X, \mathcal{J}_k^m X) = 0$$

*if  $k \geq 1$  and  $m > 2k$ .*

Actually, it was asserted in [G-G] that the preceding Theorem holds without the assumption that  $p_g(X) > 0$ . At the moment I can only get through the proof with this additional assumption.

Thus the condition of Theorem 3.14 is satisfied for a minimal surface of general type and we obtained,

**Corollary 4.5** *Let  $X$  be a smooth minimal surface of general type with  $p_g(X) > 0$  and if  $c_1^2(\mathcal{J}_k^m X) - c_2(\mathcal{J}_k^m X) > 0$  then there exists  $c > 0$  and  $m_0 > 0$  such that*

$$\dim H^0(X, \mathcal{J}_k^{k!m} X) \geq cm^{n(k+1)-1},$$

*if  $k \geq 1$  and  $m > 1$ , i. e.,  $\mathcal{J}_k^{k!} X$  is big.*

**Corollary 4.6** *Let  $X$  be a smooth minimal surface of general type with  $p_g(X) > 0$  then  $\mathcal{J}_k^{k!} X$  is big for  $k \geq 3$ .*

*Proof.* By the calculation in section 2,

$$\Delta(\mathcal{J}_3^6 X) = c_1^2(\mathcal{J}_3^6 X) - c_2(\mathcal{J}_3^6 X) = 5175c_1^2(T^*X) - 175c_2(T^*X)$$



which is clearly  $> 0$  in view of Theorem 4.1. The Corollary now follows from Corollary 4.5. QED

If  $X$  is a smooth hypersurface the preceding Corollary can be expressed in terms of the degree:

**Corollary 4.6** *Let  $X$  be a non-singular hyper surface of degree  $d$  in  $\mathbf{P}^3$ . Then  $\mathcal{J}_2^2 X$  is big if  $d \geq 9$  and  $\mathcal{J}_3^6 X$  is big if  $d \geq 5$ .*

*Proof.* By the calculation in section 2, we have:

$$c_1^2(\mathcal{J}_2^2 X) - c_2(\mathcal{J}_2^2 X) = 11c_1^2(T^*X) - 5c_2(T^*X)$$

and as noted before, for a smooth hypersurface  $X$  in  $\mathbf{P}^3$  of degree  $d$ , the Chern numbers are given by

$$c_1(T^*X) = d - 4, \quad c_2(T^*X) = d^2 - 4d + 6,$$

we conclude that:

$$c_1^2(\mathcal{J}_2^2 X) - c_2(\mathcal{J}_2^2 X) = 11d^2(d - 4)^2 - 5d^2(d^2 - 4d + 6) > 0$$

if  $d \geq 9$ . Computing similarly we conclude that  $c_1^2(\mathcal{J}_3^6 X) - c_2(\mathcal{J}_3^6 X) > 0$  if  $d \geq 5$ . Moreover, by Noether's Theorem (i. e., Riemann-Roch):

$$1 - q(X) + p_g(X) = \frac{1}{12}(c_1^2(T^*X) + c_2(T^*X))$$

implies that  $p_g(X) > 0$  because the irregularity  $q(X) = 0$ . QED

We need one last observation to deal with the fact that  $\mathcal{J}_k^m X$  is not semi-stable as can be seen from the calculation in section 2. In fact each of the factors  $\odot^{i_1} T^*X \otimes \dots \otimes \odot^{i_k} T^*X$ ,  $i_1 + 2i_2 + \dots + ki_k = m$  which is a subsheaf of  $\mathcal{J}_k^m X$  (note that not all of them are) is a destabilizing subsheaf). However, we also observe that each of these sheaves is semi-stable (by Theorem 4.3). Moreover the ratio:  $c_1(\odot^{i_1} T^*X \otimes \dots \otimes \odot^{i_k} T^*X) / \text{rank}(\odot^{i_1} T^*X \otimes \dots \otimes \odot^{i_k} T^*X) \leq m/2$  thus we have:

**Theorem 4.7** *Let  $X$  be a complex surface such that  $\text{Pic}X \cong \mathbf{Z}$ . If  $H^0(X, \mathcal{J}_k^m X \otimes [-D]) \neq \{0\}$  where  $D$  is a divisor in  $X$  then*

$$c_1([D]) \cdot c_1(T^*X) \leq \frac{m}{2} c_1^2(T^*X).$$

*Proof.* This follows from the filtration:

$$\mathcal{J}_{k-1}^m X = \mathcal{F}_k^0 \subset \mathcal{F}_k^1 \subset \dots \subset \mathcal{F}_k^{[m/k]} = \mathcal{J}_k^m X$$

(where  $[m/k]$  is the greatest integer smaller than or equal to  $m/k$ ) such that

$$\mathcal{F}_k^i / \mathcal{F}_k^{i-1} \cong \mathcal{J}_{k-1}^{m-ki} X \otimes (\odot^i T^* X).$$

From the exact sequence

$$0 \rightarrow \mathcal{F}_k^{[m/k]-1} \rightarrow \mathcal{J}_k^m X \rightarrow \mathcal{J}_{k-1}^{m-k[m/k]} X \otimes (\odot^{[m/k]} T^* X) \rightarrow 0$$

we obtain an exact sequence:

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{F}_k^{[m/k]-1} \otimes [-D]) &\rightarrow H^0(X, \mathcal{J}_k^m X \otimes [-D]) \rightarrow \\ &\rightarrow H^0(X, \mathcal{J}_{k-1}^{m-k[m/k]} X \otimes (\odot^{[m/k]} T^* X) \otimes [-D]) \end{aligned}$$

which shows that if  $H^0(X, \mathcal{J}_k^m X \otimes [-D]) \neq \{0\}$  then either  $H^0(X, \mathcal{F}_k^{[m/k]-1} \otimes [-D]) \neq \{0\}$  or

$$H^0(X, \mathcal{J}_{k-1}^{m-k[m/k]} X \otimes (\odot^{[m/k]} T^* X) \otimes [-D]) \neq \{0\}$$

and eventually this means that

$$\text{either } H^0(X, \odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X \otimes [-D]) \neq \{0\}$$

for at least one of the factors  $\odot^{i_1} T^* X \otimes \dots \otimes \odot^{i_k} T^* X$ ,  $i_1 + 2i_2 + \dots + ki_k = m$ . With this the Theorem follows from Theorem 4.3. QED

From the computation in section 2 we see that

$$\mu(\mathcal{J}_k^{k!} X) = \frac{c_1(\mathcal{J}_k^{k!} X)}{\text{rank } \mathcal{J}_k^{k!} X} < \frac{m}{2} c_1(T^* X).$$

Thus the estimate is weaker than one would get if it were stable, however this is the best that one can do and this weaker estimate is sufficient for our purpose.

We assume from now on that (i)  $X$  is a minimal surface of general type, (ii)  $p_g(X) > 0$  and (iii)  $\text{Pic}(X) \cong \mathbf{Z}$ . Then  $\mathcal{J}_k^{k!} X$  is big for  $k \geq 3$ .

This implies that  $\mathcal{J}_k^{k!}X \otimes [-D]$  is big for any effective ample divisor  $D$  in  $X$ . Schwarz Lemma implies that the lifting of any holomorphic curves in  $\mathbf{P}(J^k X)$  is contained in the zero set of a non-trivial section of  $\mathcal{O}(k!m) \otimes p^*[-D]$ . We proceed to consider the subvarieties of  $\mathbf{P}(J^k X)$ ,  $k \geq 2$ . Let  $Y_1$  be an irreducible effective horizontal (i. e., not of the form  $p^*D$  for some effective divisor  $D$  in  $X$ ) divisor in  $\mathbf{P}(J^k X)$  then:

$$[Y_1] = \mathcal{O}_{\mathbf{P}(J^k X)}(m_1) \otimes p^*[-D_1]$$

where  $D_1$  is a divisor in  $X$  and  $m_1 \in \mathbf{N}$ , we may assume that  $m_1$  is divisible by  $k!$  by replacing  $Y_1$  with  $k!Y_1$  (so it is non-reduced but set theoretically it has only one irreducible component). Thus we may write  $m_1 = k!\alpha_1$ . For simplicity of notations we shall write  $\mathcal{O}(j)$  instead of  $\mathcal{O}_{\mathbf{P}(J^k X)}(j)$ . Since  $\dim \mathbf{P}(J^k X) = 2(k+1) - 1 = 2k+1$  we get from Theorem 4.7 and Theorem 3.13:

$$\begin{aligned} c_1^{2k}(\mathcal{O}(k!)|_{Y_1}) &= c_1^{2k+1}(\mathcal{O}(k!)) \cdot (c_1(\mathcal{O}(k!\alpha_1)) - p^*c_1([D_1])) \\ &= (k!)^2 \{ \alpha_1 c_1^{2k+1}(\mathcal{O}(k!)) - c_1^{2k}(\mathcal{O}(k!)) \cdot p^*c_1([D_1]) \} \\ &= (k!)^2 \{ \alpha_1 (c_1^2(\mathcal{J}_k^{k!}X) - c_2(\mathcal{J}_k^{k!}X)) - c_1(\mathcal{J}_k^{k!}X) \cdot c_1([D_1]) \} \\ &= (k!)^2 \{ \alpha_1 (c_1^2(\mathcal{J}_k^{k!}X) - c_2(\mathcal{J}_k^{k!}X)) - a(k, k!)c_1(T^*X) \cdot c_1([D_1]) \} \\ &\geq (k!)^2 \{ \alpha_1 (a(k, k!)^2 c_1^2(T^*X) - c_2(\mathcal{J}_k^{k!}X)) - \frac{\alpha_1 k! a(k, k!)}{2} c_1^2(T^*X) \} \\ &= (k!)^2 \alpha_1 \{ a(k, k!)(a(k, k!) - \frac{k!}{2}) c_1^2(T^*X) - c_2(\mathcal{J}_k^{k!}X) \}. \end{aligned}$$

This means that  $\mathcal{O}(k!)|_{Y_1}$  is again big if

$$a(k, k!)(a(k, k!) - \frac{k!}{2}) c_1^2(T^*X) - c_2(\mathcal{J}_k^{k!}X) > 0.$$

For example if  $k = 3$ ,  $a(k, k!) = 101$  by the calculation in section 2; the preceding inequality yields:

$$\begin{aligned} c_1^{2k}(\mathcal{O}(k!)|_{Y_1}) &\geq (k!)^2 \alpha_1 \{ 101(101 - 3) c_1^2(T^*X) - c_2(\mathcal{J}_k^{k!}X) \} \\ &= (k!)^2 \alpha_1 \{ (9292 - 5026) c_1^2(T^*X) - c_2(T^*X) \} \\ &= (k!)^2 \alpha_1 (4266 c_1^2(T^*X) - 175 c_2(T^*X)) \\ &> 0. \end{aligned}$$

This means that  $\mathcal{O}(k!)|_{Y_1}$  is again big. The Schwarz Lemma in the appendix again implies that the lifting of any holomorphic curves in  $\mathbf{P}(J^k X)$  is contained in the zero set of a non-trivial section of  $\mathcal{O}(k!m)|_{Y_1} \otimes p|_{Y_1}^*[-D]$ .

Next we consider divisor  $Y_2$  in  $Y_1$  which is of the form:

$$[Y_2] = (\mathcal{O}(m_2) \otimes p^*[-D_2])|_{Y_1}$$

where  $D_2$  is a divisor in  $X$  and  $m_2 \in \mathbf{N}$  which we may assume to be divisible by  $k!$ , i.e.,  $m_2 = \alpha_2 k!$ . We remark that for the investigation of degeneration of liftings of a holomorphic curve  $f : \mathbf{C} \rightarrow X$  these are the only type of subvarieties that we have to deal with. We have:

$$\begin{aligned} & c_1^{2k-1}(\mathcal{O}(k!)|_{Y_2}) \\ &= c_1^{2k-1}(\mathcal{O}(k!)) \cdot (c_1(\mathcal{O}(k!\alpha_1) - p^*c_1([D_1]) \cdot (c_1(\mathcal{O}(k!\alpha_2) - p^*c_1([D_2])) \\ &= \alpha_1 \alpha_2 c_1^{2k+1}(\mathcal{O}(k!)) - \alpha_1 c_1^{2k}(\mathcal{O}(k!)) \cdot p^*c_1([D_2]) \\ &\quad - \alpha_2 c_1^{2k}(\mathcal{O}(k!)) \cdot p^*c_1([D_1]) + p^*c_1([D_1]) \cdot p^*c_1([D_2]) \\ &\geq (k!)^2 \{ \alpha_1 \alpha_2 (c_1^2(\mathcal{J}_k^{k!} X) - c_2(\mathcal{J}_k^{k!} X)) - \alpha_1 c_1(\mathcal{J}_k^{k!} X) \cdot c_1([D_1]) \\ &\quad - \alpha_2 \alpha_1 c_1(\mathcal{J}_k^{k!} X) \cdot c_1([D_1]) + c_1([D_1]) \cdot c_1([D_2]) \} \\ &= (k!)^2 \{ \alpha_1 \alpha_2 (c_1^2(\mathcal{J}_k^{k!} X) - c_2(\mathcal{J}_k^{k!} X)) - \alpha_1 a(k, k!) c_1(T^* X) \cdot c_1([D_2]) \\ &\quad - \alpha_2 a(k, k!) c_1(T^* X) \cdot c_1([D_1]) + c_1([D_1]) \cdot c_1([D_2]) \} \\ &\geq (k!)^2 \{ \alpha_1 \alpha_2 (c_1^2(\mathcal{J}_k^{k!} X) - c_2(\mathcal{J}_k^{k!} X)) - 2 \frac{\alpha_1 \alpha_2 k! a(k, k!)}{2} c_1^2(T^* X) \} \\ &= (k!)^2 \alpha_1 \alpha_2 \{ a(k, k!) (a(k, k!) - k!) c_1^2(T^* X) - c_2(\mathcal{J}_k^{k!} X) \}. \end{aligned}$$

Proceeding inductively, we get a sequence of subvarieties  $Y_1 \supset Y_2 \supset \dots \supset Y_{2k}$  where each  $Y_i$  is of codimension  $i$  and of the form

$$[Y_{i+1}] = (\mathcal{O}(m_{i+1}) \otimes p^*[-D_{i+1}])|_{Y_i}.$$

A similar calculation shows that:

$$\begin{aligned} & c_1^{2k-i+1}(\mathcal{O}(k!)|_{Y_i}) \\ &\geq (k!)^2 \alpha_1 \dots \alpha_i \{ a(k, k!) (a(k, k!) - k! \frac{i}{2}) c_1^2(T^* X) - c_2(\mathcal{J}_k^{k!} X) \}, \end{aligned}$$

$i = 1, \dots, 2k$ . For  $k = 3$  we have:

$$\begin{aligned} & a(3, 3!) (a(3, 3!) - 3! \frac{i}{2}) c_1^2(T^* X) - c_2(\mathcal{J}_k^{k!} X) \\ &\geq 101(101 - (3!)3) c_1^2(T^* X) - c_2(\mathcal{J}_k^{k!} X) \\ &= 3357 c_1^2(T^* X) - 175 c_2(T^* X) \\ &> 0 \end{aligned}$$

for all  $i = 1, \dots, 2k = 6$ . Thus we arrive at the following Theorem:

**Theorem 4.8** *Let  $X$  be a smooth minimal surface of general type such that (i)  $\text{Pic} X \cong \mathbf{Z}$  and (ii)  $p_g(X) > 0$ . Then  $X$  is hyperbolic. Consequently, a generic smooth hypersurface in  $\mathbf{P}^3$  of degree  $d \geq 5$  is hyperbolic.*

In [D-E] certain types of 2-jet differentials  $\mathcal{A}$  were used and the authors established that  $c_1^2(\mathcal{A}) - c_2(\mathcal{A}) = 13c_1^2(T^*X) - 9c_2(T^*X)$  on any hypersurface  $X$  of degree  $\geq 42$ . This is weaker than what we have, namely  $c_1^2(\mathcal{J}_2^2 X) - c_2(\mathcal{J}_2^2 X) = 11c_1^2(T^*X) - 5c_2(T^*X)$ . Actually I have some trouble using this stronger estimate to get hyperbolicity due to non-semistability (recall that neither  $\mathcal{J}_k^m$  nor  $T_k^*X$  is semi-stable) so that  $c_1^2(\mathcal{J}_2^2 X) - c_2(\mathcal{J}_2^2 X) = 11c_1^2(T^*X) - 5c_2(T^*X)$  is not big enough to reach hyperbolicity. It appears that  $\mathcal{A}$  is not semi-stable either.

## Appendix A: The Lemma of Logarithmic Derivatives

One of the main tool in Nevanlinna Theory is the classical Lemma of Logarithmic Derivatives (abbrev. LLD). For example, LLD implies that, even though there is no pointwise estimate between (the absolute value of) a holomorphic function and (the absolute value of) its derivatives, such estimates do exist in the sense of integral averages (i.e, their characteristic functions bound each other). The purpose of this appendix is to extend the classical LLD to all jet differentials of logarithmic type and in particular all regular jet differentials. The proof is based on the very simple observation that (the absolute value of) any jet differential of logarithmic type is bounded by (the absolute value of) those of the classical type (hence the classical LLD applies).

**Theorem A1** (Lemma of Logarithmic Derivatives) *Let  $X$  be a projective variety and let (i)  $D$  be an effective divisor with simple normal crossings or (ii)  $D$  is the trivial divisor in  $X$  (i.e. the support of  $D$  is empty or equivalently, the line bundle associate to  $D$  is trivial). Let  $f : \mathbf{C} \rightarrow X$  be a holomorphic map and  $\omega \in H^0(X, \mathcal{J}_k^m X(\log D))$  (resp.  $H^0(X, \mathcal{J}_k^m X)$  in case (ii)) a jet differential such that  $\omega \circ j^k f$  is not identically zero, then*

$$T_{\omega \circ j^k f}(r) = \int_0^{2\pi} \log^+ |\omega(j^k f(re^{\sqrt{-1}\theta}))| \frac{d\theta}{2\pi} \leq O(\log T_f(\omega_X; r)) + O(\log r).$$

Here  $\omega_X$  can be taken to be  $c_1(\mathcal{L})$  of any ample line bundle  $\mathcal{L}$  on  $X$ .

*Proof.* We claim that there exists a finite number of rational functions  $t_1, \dots, t_q$  on  $X$  such that:

(†) *the logarithmic jet differentials  $\{(d^{(j)}t_i/t_i)^{m/j} \mid 1 \leq i \leq q, 1 \leq j \leq k\}$  span the fibers of  $\mathcal{J}_k^m X(\log D)$  over every point of  $X$ .*

Without loss of generality we may assume that  $D$  is ample; otherwise we may replace  $D$  by  $D + D'$  so that  $D + D'$  is ample. Observe that if  $s$  is a function holomorphic on a neighborhood  $U$  such that  $[s = 0] = D \cap U$  then  $[s^\tau = 0] = \tau D \cap U$  where  $\tau$  is a rational number. Thus  $d^{(j)}(\log s^\tau) = \tau d^{(j)}(\log s)$  is still a jet differential with logarithmic singularity along  $D \cap U$  so the multiplicity causes no problem. This means that we may assume without loss of generality that  $D$  is very ample by replacing  $D$  with  $\tau D$  for some  $\tau$  so that  $\tau D$  is very ample.

Let  $u \in H^0(X, [D])$  be a section such that  $D = [u = 0]$ . At a point  $x \in D$  choose a section  $v_1 \in H^0(X, [D])$  so that  $E_1 = [v_1 = 0]$  is smooth,  $D + E_1$  is of simple normal crossings and  $v_1$  is non-vanishing at  $x$  (this is possible because the line bundle  $[D]$  is very ample). The rational function  $t_1 = u_1/v_1$  is regular on the affine open neighborhood  $X \setminus E_1$  of  $x$  and  $(X \setminus E_1) \cap [t_1 = 0] = (X \setminus E_1) \cap D$ . Choose rational functions  $t_2 = u_2/v_2, \dots, t_n = u_n/v_n$  where  $u_i$  and  $v_i$  are sections of a very ample bundle  $\mathcal{L}$  so that  $t_2, \dots, t_n$  are regular at  $x$ , the divisors  $D_i = [u_i = 0], E_i = [v_i = 0]$  are smooth and  $D + D_2 + \dots + D_n + E_1 + \dots + E_n$  is of simple normal crossings. Moreover, since the bundles involved are very ample the sections can be chosen so that  $dt_1 \wedge \dots \wedge dt_n$  is non-vanishing at  $x$ ; the complete system of sections provides an embedding, hence at each point there are  $n + 1$  sections with the property that  $n$  of the the quotients of these  $n + 1$  sections form a local coordinate system on some open neighborhood  $U_x$  of  $x$ . This implies that (†) is satisfied over  $U_x$ . Since  $D$  is compact it is covered by a finite number of such open neighborhoods, say  $U_1, \dots, U_p$  and a finite number of rational functions (constructed as above for each  $U_i$ ) on  $X$  so that (†) is satisfied on  $\cup_{1 \leq i \leq p} U_i$ . Moreover, there exists relatively compact open subsets  $U'_i$  of  $U_i$  ( $1 \leq i \leq p$ ) such that  $\cup_{1 \leq i \leq p} U'_i$  still covers  $D$ .

Next we consider a point  $x$  in the compact set  $X \setminus \cup_{1 \leq i \leq p} U'_i$ . Repeating the procedure as above we can find rational functions  $s_1 = a_1/b_1, \dots, s_n = a_n/b_n$  where  $a_i$  and  $b_i$  are sections of some very ample line  $\mathcal{L}$  bundle so that  $s_1, \dots, s_n$  form a holomorphic local coordinate on some open neighborhood  $V_x$  of  $x$ . Thus (†) is satisfied on  $V_x$  by the rational functions  $s_1, \dots, s_n$ . Note that we must also choose these sections so that the divisor  $H = [s_1 \dots s_n = 0]$

together with those divisors (finite in number), which had been already constructed above, is still a divisor with simple normal crossings (this is possible by the very ampleness of the line bundle  $\mathcal{L}$ .) Since  $X \setminus \cap_{1 \leq i \leq p} U'_i$  is compact, it is covered by a finite number of such coordinate neighborhoods. The coordinates are rational functions and finite in number and by construction it is clear that the condition  $(\dagger)$  is satisfied on  $X \setminus \cap_{1 \leq i \leq p} U'_i$ . Since  $\cup_{1 \leq i \leq p} U_i$  together with  $X \setminus \cap_{1 \leq i \leq p} U'_i$  covers  $X$ , the condition  $(\dagger)$  is satisfied on  $X$ .

If  $D$  is the trivial divisor, then it is enough to use only the second part of the construction above and again  $(\dagger)$  is verified with  $\mathcal{J}_k^m X(\log D) = \mathcal{J}_k^m X$

To obtain the estimate of the Theorem observe that the function,

$$\rho : J^k X(-\log D) \rightarrow [0, \infty]$$

defined by

$$\rho(\xi) = \sum_{i=1}^q \sum_{j=1}^k |(d^{(j)} t_i / t_i)^{m/j}(\xi)|^2, \quad \xi \in J^k X(-\log D)$$

( $\{t_i\}$  is the family of rational functions satisfying the condition  $(\dagger)$ ) is continuous in the extended sense; it is continuous, in the usual sense, outside the fibers over the divisor  $E$  (the sum of the divisors associated to the rational functions  $\{t_i\}$ ; note that  $E$  contains  $D$ ). Over the fiber of each point  $x \in X - E$ ,  $|(d^{(j)} t_i / t_i)^{m/j}(\xi)|^2$  is finite for  $\xi \in J^k X(-\log D)_x$ , thus  $\rho$  is not identically infinite. Moreover, since

$$\{(d^{(j)} t_i / t_i)^{m/j} \mid 1 \leq i \leq q, 1 \leq j \leq k\}$$

span the fiber of  $\mathcal{J}_k^m X(\log D)$  over every point of  $X$ ,  $\rho$  is strictly positive ( $+\infty$  allowed) outside the zero-section of  $J_k X(-\log D)$ . The quotient

$$|\omega|^2 / \rho : J^k X(-\log D) \rightarrow [0, \infty]$$

does not take on the extended value  $\infty$  when restricted to  $J^k X(-\log D) \setminus \{zero-section\}$  because, as we have just observed,  $\rho$  is non-vanishing (even though it blows up along the fibers over  $E$  so that the reciprocal  $1/\rho$  is zero there) and the singularity of  $|\omega|$  is no worse than that of  $\rho$  because the singularity of  $\omega$  occurs only along  $D$  (which is contained in  $E$ ) and is of log type. Thus the restriction to  $J_k X(-\log D) \setminus \{zero-section\}$ ,

$$|\omega|^2 / \rho : J^k X(-\log D) \setminus \{zero-section\} \rightarrow [0, \infty)$$

is a continuous non-negative function. Moreover, since  $|\omega|$  and  $\rho$  have the same homogeneity:

$$|\omega(\lambda.\xi)|^2 = |\lambda|^{2m} |\omega(\lambda.\xi)|^2 \text{ and } \rho(\lambda.\xi) = |\lambda|^{2m} \rho(\xi)$$

for all  $\lambda \in \mathbf{C}^*$  and  $\xi \in J^k X(-\log D)$  we see that  $|\omega|^2/\rho$  descends to a well-defined function on  $\mathbf{P}(E_{k,D}) = (J^k X(-\log D) \setminus \{\text{zero-section}\})/\mathbf{C}^*$ , i.e.,

$$|\omega|^2/\rho : \mathbf{P}(E_{k,D}) \rightarrow [0, \infty)$$

is a well-defined continuous function and so, by compactness, there exists a constant  $c$  with the property that

$$|\omega|^2 \leq c\rho.$$

This implies that

$$\begin{aligned} T_{\omega \circ j^k f}(r) &= \int_0^{2\pi} \log^+ |\omega(j^k f(re^{\sqrt{-1}\theta}))| \frac{d\theta}{2\pi} \\ &\leq \int_0^{2\pi} \log^+ |\rho(j^k f(re^{\sqrt{-1}\theta}))| \frac{d\theta}{2\pi} + O(1). \end{aligned}$$

Since  $t_i$  is a rational function on  $X$  the function

$$(d^{(j)} t_i / t_i)^{m/j} (j^k f) = ((t_i \circ f)^{(j)} / t_i \circ f)^{m/j}$$

( $m$  is divisible by  $k$ !) is meromorphic on  $\mathbf{C}$  and so, by the definition of  $\rho$ ,

$$\log^+ |\rho(j^k f)| \leq O(\max_{1 \leq i \leq q, 1 \leq j \leq k} \log^+ |(t_i \circ f)^{(j)} / t_i \circ f|) + O(1).$$

Now by the classical lemma of logarithmic derivatives for meromorphic functions,

$$\int_0^{2\pi} \log^+ |(t_i \circ f)^{(j)} / t_i \circ f| \frac{d\theta}{2\pi} \leq O(\log r T_{t_i \circ f}(r)).$$

Since  $t_i$  is a rational function,

$$\log T_{t_i \circ f}(r) \leq O(\log T_f(\omega_X; r)) + O(1)$$

and we arrive at the estimate

$$\begin{aligned} &\int_0^{2\pi} \log^+ |\rho(j^k f(re^{\sqrt{-1}\theta}))| \frac{d\theta}{2\pi} \\ &\leq O(\int_0^{2\pi} \log^+ |(t_i \circ f)^{(j)} / t_i \circ f| \frac{d\theta}{2\pi}) + O(1) \\ &\leq O(\log T_f(r)) + O(\log r). \end{aligned}$$



This implies that

$$T_{\omega \circ j^k f}(r) \leq O(\log T_f(r)) + O(\log r)$$

as claimed. QED

We obtain, as immediate consequence, the following Schwarz's type Lemma for logarithmic jet differentials.

**Corollary A2** *Let  $X$  be a projective variety and  $D$  be an effective divisor (possibly the trivial divisor) with simple normal crossings. Let  $f : \mathbf{C} \rightarrow X \setminus D$  be a holomorphic map. Then*

$$\omega(j^k f) \equiv 0 \quad \text{for all } \omega \in H^0(X, \mathcal{J}_k^m X(\log D) \otimes [-H])$$

where  $H$  is a generic hyperplane section (and hence any hyperplane section).

*Proof.* If  $f$  is constant then the Corollary holds trivially. So we may assume that  $f$  is non-constant and suppose that  $\omega \circ j^k f \not\equiv 0$ . Moreover, since  $f$  is non-constant, we may assume without loss of generality that  $\log r = o(T_f(H; r))$  by replacing  $f$  with  $f \circ \phi$  where  $\phi$  is a transcendental function on  $\mathbf{C}$ . By Theorem 4.1, we have

$$\int_0^{2\pi} \log^+ |\omega \circ j^k f| \frac{d\theta}{2\pi} = T_{\omega \circ j^k f}(r) \leq O(\log r T_f(H; r)).$$

On the other hand, since  $\omega$  vanishes on  $H$  and  $H$  is generic (see (1) or (2) in section 1), we obtain via Jensen's Formula:

$$\begin{aligned} T_f(H; r) &\leq N_f(H; r) + O(\log r T_f(H; r)) \\ &= \int_0^{2\pi} \log |\omega \circ j^k f| \frac{d\theta}{2\pi} + O(\log r T_f(H; r)) \end{aligned}$$

which, together with the preceding estimate, imply that:

$$T_f(H; r) \leq O(\log r T_f(H; r)).$$

This is impossible hence we must have  $\omega \circ j^k f \equiv 0$ . If  $H_1 = [s_1 = 0]$  is any hyperplane section then it is linearly equivalent to a generic hyperplane section  $H = [s = 0]$ . If  $\omega$  vanishes along  $H'$  then  $(s/s_1)\omega$  vanishes along  $H$ . The preceding discussion implies that  $(s/s_1)\omega(j^k f) \equiv 0$ . This implies that actually  $\omega(j^k f) \equiv 0$  as we may choose a generic section  $H$  so that the image of  $f$  is not entirely contained in  $H$ . QED

Actually the proof of Theorem A.1 gives a little more. In fact the same proof yields:

**Theorem A3** *Let  $\rho_k$  be a pseudo singular jet metric on  $J^k X(-\log D)$  with the property that there exists a constant  $c > 0$  such that  $\rho_k \leq \rho$  where  $\rho$  is the singular jet metric on  $J^k X(-\log D)$  defined by the family of rational functions (†) (see (28)). Then*

$$T_{j^k f}(\rho_k; r) = \int_0^{2\pi} \log^+ |\rho_k(j^k f(re^{\sqrt{-1}\theta}))| \frac{d\theta}{2\pi} \leq O(\log r T_f(\omega_X; r))$$

for any Kähler metric  $\omega_X$  on  $X$ . In particular, if  $\rho_k$  is a non-singular pseudo metric on  $J^k X$  then the preceding estimate holds.

The Schwarz Lemma can be further extended as follows.

**Theorem A4** *Let  $Y \subset \mathbf{P}(J^k X)$  be a subvariety and suppose that there exists a non-trivial section  $\sigma \in H^0(Y, \mathcal{O}_{\mathbf{P}(J^k X)}(m)|_Y \otimes p_Y^*[-D])$  where  $D$  is a generic ample divisor in  $X$  and  $p : \mathbf{P}(J^k X) \rightarrow X$  is the projection map. If the image of the lifting  $[j^k f] : \mathbf{C} \rightarrow \mathbf{P}(J^k X)$  of a holomorphic curve  $f : \mathbf{C} \rightarrow X$  is contained in  $Y$  then  $\sigma([j^k f]) \equiv 0$ .*

## Appendix B

Let  $S_n$  be the symmetric group on  $n$  elements then the order of  $S_n$  is  $n!$ .

**Definition B1** A maximal set of mutually conjugate elements of  $S_n$  is said to be a class of  $S_n$ .

**Definition B2** A partition of a natural number  $n$  is a set of positive integers  $k_1, \dots, k_q$  such that  $n = k_1 + \dots + k_q$ .

A partition can be expressed as

$$n = \sum_{i=1}^n i a_i$$

where the integers  $a_i$  are non-negative.

**Theorem B3** *The number, denoted  $p(n)$ , of classes of  $S_n$  is equal to the number of partitions of  $n$  and also to the number of inequivalent irreducible*

representations of  $S_n$ . The number  $p(n)$  is asymptotically approximated by the formula of Hardy-Ramanujan

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

The alternating subgroup  $A_n$  (i.e. the even permutations) is the commutator subgroup of  $S_n$  and is obviously of index 2. Thus there are two 1-dimensional representations of  $S_n$ : the trivial representation and the representation  $\Gamma_\sigma$  defined by  $P \mapsto \sigma(P)$  where  $\sigma$  is the signature of a permutation  $P$  (i.e.  $P \mapsto \pm 1$  depending on whether  $P$  is even or odd (i.e., can be expressed as an even or odd number of transpositions: interchanging two of the  $n$  elements)).

**Lemma B4** *Let  $X = X(n)$  be a set of  $n$  elements and let  $Y_1, \dots, Y_k$  be  $k$  not necessarily distinct subsets of  $X$ . For any subset  $J$  of the index set  $\{1, \dots, k\}$ , denote by*

$$n(J) = \# \cap_{j \in J} Y_j;$$

*and for  $0 \leq i \leq k$ , denote by*

$$n_0 = n, \quad n_i = \sum_{\#J=i} n(J), \quad 1 \leq i \leq k.$$

*Then the number of elements not in any of the subsets  $Y_i, i = 1, \dots, k$  is given by the formula*

$$\#(X \setminus (\cup_{i=1}^k Y_i)) = n - n_1 + n_2 - \dots + (-1)^k n_k = \sum_{i=0}^k (-1)^i n_i.$$

*Proof.* If  $k = 2$  the formula can be expressed as usual:

$$\#(X \setminus (Y_1 \cup Y_2)) = \#X - \#Y_1 - \#Y_2 + \#(Y_1 \cap Y_2).$$

One way to prove the Lemma is by induction on  $k$ . Alternatively one can also argue as follows: QED

An element  $P$  of  $S_n$  is said to be a derangement if  $P(i) \neq i$  for  $i = 1, \dots, n$ . The number of derangements is denoted by  $d_n$ . Then

**Corollary B5** *The number of derangements is given by the formula*

$$d_n = n! \sum_{i=0}^n \frac{(-1)^i}{i!}.$$

*In particular we see that asymptotically  $d_n \sim e^{-1}$ .*

*Proof.* Apply Lemma 4 with  $X = \mathcal{S}_n$  and  $Y_i = \{P \in \mathcal{S}_n \mid P(i) = i\}, i = 1, \dots, n$ .

Alternatively, the formula can be obtained by considering the power series:

$$e^x \sum_{i=0}^{\infty} d_i \frac{x^i}{i!} = \sum_{i=0}^{\infty} \left( \sum_{j=0}^i \frac{i!}{j!(i-j)!} d_{i-j} \right) \frac{x^i}{i!} = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$

Thus we have

$$\sum_{i=0}^{\infty} d_i \frac{x^i}{i!} = e^{-x}(1-x)^{-1}$$

which yields the formula of the corollary. QED

The formula of the Corollary can also be obtained via the recursive formula:

$$d_n = nd_{n-1} + (-1)^n.$$

**Corollary B6** *The number of surjections from a set  $A$  of  $n$  elements to a set  $B$  of  $k$  elements is given by the formula*

$$\sum_{i=0}^k (-1)^i \frac{k!}{i!(k-i)!} (k-i)^n.$$

*Proof.* Apply Lemma 4 to the set  $X$  of all maps from  $A$  to  $B$  and  $Y_i, i = 1, \dots, k$  be the subset consisting of those maps such that  $i$  is not in the image. QED

Note that Corollary 6 implies trivially that

$$\sum_{i=0}^k (-1)^i \frac{k!}{i!(k-i)!} (k-i)^n = \begin{cases} n! & \text{if } k = n, \\ 0 & \text{if } k > n. \end{cases}$$

There is a more general formula which can be proved in a similar fashion:

$$\sum_{i=0}^n (-1)^i \frac{n!}{i!(n-i)!} \frac{(m+n-i)!}{(m+n-k)!(k-i)!} = \begin{cases} m!/k!(m-k)! & \text{if } m \geq k, \\ 0 & \text{if } m < k. \end{cases}$$

**Theorem B7** *The number of non-negative integer solutions of the equation*

$$x_1 + \dots + x_k = n$$

*is  $(n+k-1)!/(k-1)!n!$ . On the other hand the number of positive integer solutions is  $(n-1)!/(k-1)!(n-k)!$ .*

*Proof.* So we have to find the number of ways to put  $n$  *black* (otherwise identical) balls in  $k$  slots. If we insert white balls in between the slots we end up with a total of  $n+k-1$  balls  $k-1$  of them white. This is the same as choosing  $k-1$  balls from a total of  $n+k-1$  balls and the first assertion follows.

The second assertion follows from the first by making the substitution  $y_i = x_i - 1$ . resulting in the equation

$$y_1 + \dots + y_k = n - k.$$

QED

The number  $(n+k-1)!/(k-1)!n!$  is the coefficient of  $x^n$  in the expansion of the function

$$\frac{1}{(1-x)^k} = \sum_{i=0}^{\infty} c_n x^n. \quad (33)$$

Let  $c_{n,k}$  be the number of elements of  $\mathcal{S}_n$  consisting of exactly  $k$  cycles.

**Theorem B8** *With the notations above we have*

$$c_{n,k} = (n-1)c_{n-1,k} + c_{n-1,k-1}$$

*and these numbers are the coefficients of the expansion of the function  $x(x+1)\dots(x+n-1)$*

$$x(x+1)\dots(x+n-1) = \sum_{k=0}^n c_{n,k} x^k$$

and also

$$\frac{x!}{(x-n)!} = \sum_{k=0}^n (-1)^{n-k} c_{n,k} x^k.$$

Moreover these numbers are the coefficients of the expansion of the function

$$\log(1+x)^k = k! \sum_{n=k}^{\infty} c_{n,k} \frac{x^n}{n!}.$$

*Proof.* The recursive relation follows from the observation that there are exactly  $n-1$  different ways to get a permutation on  $n$  elements consisting of exactly  $k$  cycles from a permutation on  $n-1$  elements consisting of exactly  $k$  cycles. These account for the first term on the right of the recursive formula. Next we observe that there is exactly one way to get a permutation on  $n$  elements consisting of exactly  $k$  cycles from a permutation on  $n-1$  elements consisting of exactly  $k-1$  cycles and these account for the second term in the formula. The rest of the Theorem follows from the observation that if we write

$$x(x+1)\dots(x+n-1) = \sum_{k=0}^n a_{n,k} x^k$$

then the coefficients satisfy the same recursive formula as  $c_{n,k}$ :

$$a_{n,k} = (n-1)a_{n-1,k} + a_{n-1,k-1}.$$

The last assetion follows by observing that

$$(1+x)^t = e^{t \log(1+x)} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k (\log(1+x))^k.$$

On the other hand, we have

$$\begin{aligned} (1+x)^t &= \sum_{n=0}^{\infty} \frac{t!}{n!(t-n)!} x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{j=0}^n c_{n,j} t^j \\ &= \sum_{j=0}^{\infty} t^j \sum_{n=j}^{\infty} c_{n,j} \frac{x^n}{n!}. \end{aligned}$$

QED

Denote by  $P(n, k)$  the set of all partitions of a set of  $n$  elements into  $k$  non-empty subsets and let

$$p_{n,k} = \#P(n, k).$$

**Theorem B9** *With the notations above we have*

$$p_{n,k} = kp_{n-1,k} + p_{n-1,k-1}$$

*and these numbers are the coefficients of the expansion of the function*

$$x^n = \sum_{k=0}^n p_{n,k} \frac{x!}{(x-k)!}$$

*and also as the coefficients of the power series expansion*

$$(e^x - 1)^k = k! \sum_{n=k}^{\infty} p_{n,k} \frac{x^n}{n!}.$$

*Proof.* The recursive formula follows from the observation that a partition of  $n$  elements into  $k$  subsets can be obtained from a partition of  $n-1$  elements into  $k$  subsets by inserting the element  $n$  into any one of the  $k$  subsets. Alternatively one can also get a partition of  $n$  elements into  $k$  subsets from a partition of  $n-1$  elements into  $k-1$  subsets by simply adding one more subset consisting of just the element  $n$ .

For a positive integer  $x$  there are exactly  $x^n$  maps from the set  $\{1, \dots, n\}$  of  $n$  elements to the set  $\{1, \dots, x\}$ . On the other hand, by definition of the number  $p_{n,k}$  we have the relation:

$$k!p_{n,k} = \# \text{ of surjections from a set of } n \text{ element onto a set of } k \text{ elements.}$$

Hence for any subset  $Y$  of  $k$  elements of  $\{1, \dots, x\}$  there are  $k!p_{n,k}$  surjections from  $\{1, \dots, n\}$  onto the set  $Y$ . Since the number of subsets of  $k$  elements of  $\{1, \dots, x\}$  is  $x!/k!(x-k)!$  we get

$$x^n = \sum_{k=0}^n \frac{x!}{k!(x-k)!} k!p_{n,k} = \sum_{k=0}^n \frac{x!}{(x-k)!} p_{n,k}.$$

By Corollary 4 we have:

$$k!p_{n,k} = \sum_{i=0}^k (-1)^i \frac{k!}{i!(k-i)!} (k-i)^n = \sum_{i=1}^k (-1)^{k-i} \frac{k!}{i!(k-i)!} i^n. \quad (34)$$

If  $k = 1$  then  $p_{n,1} = 1$  as there is only one such partition. The usual expansion of the exponential function yields

$$e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

The case of general  $k$  can be verified by induction by differentiating the power series

$$F_k(x) = \sum_{n=k}^{\infty} s_{n,k} \frac{x^n}{n!}$$

resulting in

$$\begin{aligned} F'_k(x) &= \sum_{n=k}^{\infty} p_{n,k} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=k}^{\infty} (kp_{n-1,k} + p_{n-1,k-1}) \frac{x^{n-1}}{(n-1)!} \\ &= kF_k(x) + F_{k-1}(x). \end{aligned}$$

By induction hypothesis we have:

$$F_{k-1}(x) = \frac{1}{(k-1)!} (e^x - 1)^{k-1}$$

hence the function  $F_k$  satisfies the differential equation

$$F'_k(x) = kF_k(x) + \frac{1}{(k-1)!} (e^x - 1)^{k-1}.$$

It is clear that

$$F_k(x) = \frac{1}{k!} (e^x - 1)^k$$

is a solution and is indeed the unique solution satisfying  $p_{k,k} = 1$ . QED

**Theorem B10** *The number of partitions of  $n$*

$$p(n) = \sum_{k=1}^n \frac{k^n}{k!}.$$

Denote by  $p_k(n)$  the number of solutions of the equation

$$x_1 + \dots + x_k = n \tag{35}$$



with the condition that  $1 \leq x_k \leq x_{k-1} \leq \dots \leq x_1$ . This number is obviously equal to the number of solutions of the equation

$$y_1 + \dots + y_k = n - k \quad (36)$$

with the condition that  $0 \leq y_k \leq y_{k-1} \leq \dots \leq y_1$ . If there are exactly  $i$  of the integers  $y_i$  which are positive then these are the solutions of  $x_1 + \dots + x_i = n - k$  and so there are  $p_i(n - k)$  of such solutions; consequently we have:

**Theorem B11** *With the notations above we have*

$$p_k(n) = \sum_{i=1}^k p_i(n - k).$$

Consider the case  $k = 3$  then the number of solutions of

$$x_1 + x_2 + x_3 = n$$

such that  $0 \leq x_3 \leq x_2 \leq x_1$  is the same as  $p_3(n + 3)$ . Let  $y_1 = x_1 - x_2 \geq 0, y_2 = x_2 - x_3 \geq 0, y_3 = x_3 \geq 0$  then this is also the number of solutions of the equation

$$y_1 + 2y_2 + 3y_3 = n$$

with the condition that  $y_i \geq 0$ . Thus the number  $p_3(n + 3)$  is the coefficient of  $x^n$  in the expansion of the function (compare ())

$$(1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} = \sum_{n=0}^{\infty} p_3(n + 3) x^n.$$

We have the factorization

$$(1 - x^3) = (1 - x)(1 - \theta x)(1 - \theta^2 x)$$

where  $\theta$  is a 3-rd root of unity, hence

$$\begin{aligned} & (1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \\ &= (1 - x)^{-3}(1 + x)^{-1}(1 - \theta x)^{-1}(1 - \theta^2 x)^{-1} \\ &= \frac{1}{6(1 - x)^3} + \frac{1}{4(1 - x)^2} + \frac{17}{72(1 - x)} + \frac{1}{8(1 + x)} + \\ & \quad + \frac{1}{9(1 - \theta x)} + \frac{1}{9(1 - \theta^2 x)} \end{aligned}$$

and we get from the expansion of each of the term of the partial fraction decomposition that

$$p_3(n+3) = \frac{(n+3)^2}{12} - \frac{7}{72} + \frac{(-1)^n}{8} + \frac{\theta^n + \theta^{2n}}{9}.$$

We infer that

$$|p_3(n+3) - \frac{(n+3)^2}{12}| < \frac{1}{2}$$

or equivalently that

$$|p_3(n) - \frac{n^2}{12}| < \frac{1}{2}.$$

The following identity is easily established by induction:

**Theorem B12** *The number  $p_k(n)$  satisfies the following recursive relation:  $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$ .*

Obviously we have  $p_1(n) = n$  and  $p_2(n) = n/2$  or  $(n-1)/2$  according to  $n$  being even or odd. Thus Theorem 10 yields  $p_3(n) = p_2(n-1) + p_3(n-3)$ ,  $p_4(n) = p_3(n-1) + p_4(n-4)$ ,  $p_5(n) = p_4(n-1) + p_5(n-5)$  and we get for example

$$p_1(6) = 1, p_2(6) = 3, p_6(6) = 1$$

$$p_3(6) = p_2(5) + p_3(3) = 3,$$

$$p_4(6) = p_3(5) = p_2(4) = 2,$$

$$p_5(6) = p_4(5) = p_3(4) = p_2(3) = 1$$

hence

$$p(6) = \sum_{k=1}^6 p_k(6) = 1 + 3 + 3 + 2 + 1 + 1 = 11.$$

The total partition length  $L(n)$  of a positive integer  $n$  is defined to be

$$L(n) = \sum_{k=1}^n kp_k(n). \quad (37)$$

For example if  $n = 6$  then  $L(6) = 1 + 6 + 9 + 8 + 5 + 6 = 35$ .

For  $n = 7$  we have

$$\begin{aligned}
p_1(7) &= 1, p_2(7) = 3, p_7(7) = 1 \\
p_3(7) &= p_2(6) + p_3(4) = p_2(6) + p_2(3) = 4, \\
p_4(7) &= p_3(6) = 3, \\
p_5(7) &= p_4(6) = 2, \\
p_6(7) &= p_5(6) = 1
\end{aligned}$$

hence

$$p(7) = \sum_{k=1}^7 p_k(7) = 1 + 3 + 4 + 3 + 2 + 1 + 1 = 15$$

and the total partition length

$$L(7) = 1 + 6 + 12 + 12 + 10 + 6 + 7 = 54.$$

For general  $k$  one has the following asymptotic formula:

**Theorem B13** *For  $n \rightarrow \infty$  the number  $p_k(n)$  is asymptotically given by:*

$$p_k(n) \sim \frac{n^{k-1}}{(k-1)!k!}.$$

*Proof.* The number  $p_k(n)$  is defined to be the number of solutions of  $x_1 + \dots + x_k = n$  with the condition that  $1 \leq x_k \leq x_{k-1} \leq \dots \leq x_1$ . If we drop this last condition then the  $k!$  permutations of a solution is also a solution of  $x_1 + \dots + x_k = n$ . However since  $x_i$  may equal  $x_j$  for  $i \neq j$  hence we have the inequality:

$$C_{k-1}^{n-1} = \frac{(n-1)!}{(k-1)!(n-k)!} \leq k! p_k(n).$$

On the other hand, if we set  $y_i = x_i + (k-i)$  and if  $x_1, \dots, x_k$  is a solution with  $1 \leq x_k \leq x_{k-1} \leq \dots \leq x_1$  then the  $y_i$ 's are distinct and is a solution of the equation:

$$y_1 + \dots + y_k = n + \frac{k(k-1)}{2}.$$

From this we obtain a reverse inequality:

$$k! p_k(n) \leq C_{k-1}^{n+(k(k-1)/2)-1} = \frac{\{n + (k(k-1)/2) - 1\}!}{(k-1)!\{n + k(k-1)/2 - k\}!}.$$

The Theorem follows immediately from these two estimates. QED

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